Section 9.5 Alternating Series Test

Example 1

Use Alternating Series Test to show $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n+8}$ converges.

a) Let $a_n = \frac{1}{n+8}$ and show that $a_n = \frac{1}{n+8}$ is decreasing.

Note: $a_n = \frac{1}{n+8}$ and $a_{n+1} = \frac{1}{n+9}$; therefore, $a_{n+1} < a_n$

b) $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{n+8} = 0$

Therefore, $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n+8}$ converges by the Alternating Series Test.

Example 2

Use Alternating Series Test to show $\sum_{n=1}^{\infty} \frac{(-1)^n}{2^n}$ converges.

a) Let $a_n = \frac{1}{2^n}$ and show that $a_n = \frac{1}{2^n}$ is decreasing.

$a_n = \frac{1}{2^n}$ an $a_{n+1} = \frac{1}{2^{n+1}}$; therefore $a_{n+1} < a_n$

b) $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{2^n} = 0$

Therefore $\sum_{n=1}^{\infty} \frac{(-1)^n}{2^n}$ converges by the Alternating Series Test.

Example 3

Use *n*th Term Test to show $\sum_{n=1}^{\infty} \frac{(-1)^n (3n-1)}{(n+1)}$ diverges.

Let $a_n = \frac{(3n-1)}{(n+1)}$.

$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{(3n-1)}{(n+1)} = 3$ Using L'Hopital's Rule.

The *n*th Term Test states that if $\lim_{n\to\infty} a_n \neq 0,$ then the series diverges.

Therefore, $\sum_{n=1}^{\infty} \frac{(-1)^n (3n-1)}{(n+1)}$ diverges.

Example 4

Use Alternating Series Test to show $\sum_{n=1}^{\infty} \frac{(-1)^n}{n!}$ converges.

a) Let $a_n = \frac{1}{n!}$ and $a_{n+1} = \frac{1}{(n+1)!}$; therefore $a_{n+1} < a_n$

b) $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{n!} = 0$

Hence, $\sum_{n=1}^{\infty} \frac{(-1)^n}{n!}$ converges by the Alternating Series Test.

Example 5

Use Alternating Series Test to show $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}\sqrt{n}}{n+2}$ converges.

Let $a_n = \frac{\sqrt{n}}{n+2}$ and $a_{n+1} = \frac{\sqrt{n+1}}{(n+1)+2} = \frac{\sqrt{n+1}}{n+3}$;

To see that $a_{n+1} < a_n$, we can look at the graphs

$y = \frac{\sqrt{x}}{x+2}$ and $y = \frac{\sqrt{x+1}}{x+3}$.

We can see that graph of $y = \frac{\sqrt{x+1}}{x+3}$ is below the

graph of $y = \frac{\sqrt{x}}{x+2}$ for $x \geq 2$, therefore $a_{n+1} < a_n$.

(Also, another way to show $a_{n+1} < a_n$ is by showing

the derivative $D_x\left(\frac{\sqrt{x}}{x+2}\right) < 0$ for $x \geq 2$)

Find $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{\sqrt{n}}{n+2} = 0$ Using L'Hopital's Rule

Hence, $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}\sqrt{n}}{n+2}$ converges by Alternating Series Test.

Example 6

Show that $\sum_{n=1}^{\infty}\frac{(-1)^n}{4^n}$ converges absolutely.

First we need to show $\sum_{n=1}^{\infty}\frac{(-1)^n}{4^n}$ converges by Alternating Series Test.

Let $a_n = \frac{1}{4^n}$; and $a_n = \frac{1}{4^{n+1}}$; hence, $a_{n+1} < a_n$

$\lim_{n\to\infty} a_n = \lim_{n\to\infty}\frac{1}{4^n} = 0.$

Hence, $\sum_{n=1}^{\infty}\frac{(-1)^n}{4^n}$ converges by Alternating Series Test.

Next we need to show $\sum_{n=1}^{\infty}\left|\frac{(-1)^n}{4^n}\right| = \sum_{n=1}^{\infty}\left(\frac{1}{4}\right)^n$ converges.

The series $\sum_{n=1}^{\infty}\left|\frac{(-1)^n}{4^n}\right| = \sum_{n=1}^{\infty}\left(\frac{1}{4}\right)^n$ is a Geometric Series with $r = \frac{1}{4}$;

Therefore, $\sum_{n=1}^{\infty}\left|\frac{(-1)^n}{4^n}\right| = \sum_{n=1}^{\infty}\left(\frac{1}{4}\right)^n$ converges.converges by Geometric Series Test

Summary: $\sum_{n=1}^{\infty}\frac{(-1)^n}{4^n}$ converges; and $\sum_{n=1}^{\infty}\left|\frac{(-1)^n}{4^n}\right|$ converges.

Therefore, $\sum_{n=1}^{\infty}\frac{(-1)^n}{4^n}$ converges absolutely.

Example 7

Show that $\sum_{n=1}^{\infty} \frac{(-1)^n}{(n+1)!}$ converges absolutely.

First show that $\sum_{n=1}^{\infty} \frac{(-1)^n}{n!}$ converges by Alternating Series Test.

Let $a_n = \frac{1}{(n+1)!}$; $a_{n+1} = \frac{1}{(n+2)!}$; hence, $a_{n+1} < a_n$.

$\lim_{n \to \infty} a_n = \lim_{x \to \infty} \frac{1}{(n+1)!}$

Hence, $\sum_{n=1}^{\infty} \frac{(-1)^n}{(n+1)!}$ converges by Alternating Series Test.

Next show that $\sum_{n=1}^{\infty} \left| \frac{(-1)^n}{(n+1)!} \right| = \sum_{n=1}^{\infty} \frac{1}{(n+1)!}$ converges.

Earlier we have shown that $\sum_{n=1}^{\infty} \frac{1}{(n+1)!}$ converges

by Limit Comparison Test.

In summary, $\sum_{n=1}^{\infty} \frac{(-1)^n}{(n+1)!}$ converges absolutely

because $\sum_{n=1}^{\infty} \frac{(-1)^n}{(n+1)!}$ converges and

$\sum_{n=1}^{\infty} \left| \frac{(-1)^n}{(n+1)!} \right| = \sum_{n=1}^{\infty} \frac{1}{(n+1)!}$ also converges.

Example 8

Show that $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\sqrt[3]{n}}$ converges conditionally.

First show that $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\sqrt[3]{n}}$ converges by Alternating Series Test.

Let $a_n = \frac{1}{\sqrt[3]{n}}$; and $a_{n+1} = \frac{1}{\sqrt[3]{n+1}}$; hence, $a_{n+1} < a_n$.

$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{\sqrt[3]{n}} = 0.$

Hence, $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\sqrt[3]{n}}$ converges by Alternating Series Test.

Next show that $\sum_{n=1}^{\infty} \left| \frac{(-1)^{n+1}}{\sqrt[3]{n}} \right| = \sum_{n=1}^{\infty} \frac{1}{n^{1/3}}$ diverges.

$\sum_{n=1}^{\infty} \frac{1}{n^{1/3}}$ is a p-series with $p = \frac{1}{3}$; therefore it diverges.

In summary, $\sum_{n=1}^{\infty} \frac{(-1)^{n}}{\sqrt[3]{n}}$ converges conditionally because

$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\sqrt[3]{n}}$ converges but $\sum_{n=1}^{\infty} \left| \frac{(-1)^{n+1}}{\sqrt[3]{n}} \right| = \sum_{n=1}^{\infty} \frac{1}{n^{1/3}}$ diverges.

Example 9

Show that the series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1} n^2}{(n+5)^2}$ diverges.

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n^2}{(n+5)^2} = \lim_{n\to\infty} \frac{n^2}{n^2+10n+25} = 1$$

Hint: Use L'Hopital's Rule

The *n*th Term Test states that if $\lim_{n\to\infty} a_n \neq 0,$ then series diverges. Therefore, $\sum_{n=1}^{\infty} \frac{(-1)^{n+1} n^2}{(n+5)^2}$ diverges.

Example 10

Show that $\sum_{n=1}^{\infty} \frac{(-1)^n}{(2n+1)!}$ converges absolutely.

First show that $\sum_{n=1}^{\infty} \frac{(-1)^n}{(2n+1)!}$ converges by Alternating Series Test.

Let $a_n = \frac{1}{(2n+1)!}$; and $a_{n+1} = \frac{1}{(2\left(n+1\right)+1)!} = \frac{1}{(2n+3)!}$

$\lim_{x\to\infty} a_n = \lim_{x\to\infty} \frac{1}{(2n+1)!} = 0$

Hence $\sum_{n=1}^{\infty} \frac{(-1)^n}{(2n+1)!}$ converges by Alternating Series Test.

Next show that $\sum_{n=1}^{\infty} \left| \frac{(-1)^n}{(2n+1)!} \right| = \sum_{n=1}^{\infty} \frac{1}{(2n+1)!}$ converges.

Earlier we have shown that $\sum_{n=1}^{\infty} \left| \frac{(-1)^n}{(2n+1)!} \right| = \sum_{n=1}^{\infty} \frac{1}{(2n+1)!}$ converges by Limit Comparison Test.

In summary, $\sum_{n=1}^{\infty} \frac{(-1)^n}{(2n+1)!}$ converges absolutely because $\sum_{n=1}^{\infty} \frac{(-1)^n}{(2n+1)!}$ converges and $\sum_{n=1}^{\infty} \left| \frac{(-1)^n}{(2n+1)!} \right| = \sum_{n=1}^{\infty} \frac{1}{(2n+1)!}$ also converges.