Section 9.6 Ratio Test and Root Test

Ratio Test

If $\lim_{x\to\infty}\left(\frac{a_{n+1}}{a_n}\right)<1$, then the series $\sum_{n=1}^{\infty} a_n$ converges.

If $\lim_{x\to\infty}\left(\frac{a_{n+1}}{a_n}\right)>1$, then the series $\sum_{n=1}^{\infty} a_n$ diverges.

If $\lim_{x\to\infty}\left(\frac{a_{n+1}}{a_n}\right)=1$, use another test.

Show that $\sum_{\text{n=1}}^{\infty} \frac{1}{3^n}$ converges by Ratio Test.

Let: $a_n = \frac{1}{3^n}$ and $a_{n+1} = \frac{1}{3^{n+1}}$

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{\frac{1}{3^{n+1}}}{\frac{1}{3^n}}\right| = \lim_{n\to\infty} \left|\frac{3^n}{3^{n+1}}\right| = \lim_{n\to\infty} \left|\frac{3^n}{3^n \cdot 3^1}\right| = \frac{1}{3} < 1$$

Therefore $\sum_{n=1}^{\infty} \frac{1}{3^n}$ converges by Ratio Test.

Show that $\sum_{\text{n=1}}^{\infty} \frac{n!}{5^n}$ diverges by Ratio Test.

Let $a_n = \frac{n!}{5^n}$ and $a_{n+1} = \frac{(n+1)!}{5^{n+1}}$

Note: $(n+1)! = (n+1)(n)(n-1)\cdots(3)(2)(1);$

$$n! = (n)(n-1)\cdots(3)(2)(1)$$

$$\frac{(n+1)!}{n!} = n+1$$

$$\frac{5^n}{5^{n+1}} = \frac{5^n}{5^n \cdot 5^1} = \frac{1}{5}$$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\frac{(n+1)!}{5^{n+1}}}{\frac{n!}{5^n}}\right| = \lim_{n\to\infty}\left|\frac{(n+1)!}{n!}\cdot\frac{5^n}{5^{n+1}}\right| = \lim_{n\to\infty}\left|(n+1)\left(\frac{1}{5}\right)\right| = \infty$$

Therefore $\sum_{\text{n=1}}^{\infty} \frac{n!}{5^n}$ diverges by Ratio Test.

Show that $\sum_{n=1}^{\infty} n\left(\frac{8}{5}\right)^{n}$ diverges by Ratio Test.

$$a_n = n\left(\frac{8}{5}\right)^{n} \quad and \quad a_{n+1} = (n+1)\left(\frac{8}{5}\right)^{n+1}$$

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)\left(\frac{8}{5}\right)^{n+1}}{n\left(\frac{8}{5}\right)^{n}} = \frac{(n+1)}{n}\frac{\left(\frac{8}{5}\right)^{n}\left(\frac{8}{5}\right)^{1}}{\left(\frac{8}{5}\right)^{n}} = \frac{(n+1)}{n}\left(\frac{8}{5}\right)$$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)\left(\frac{8}{5}\right)^{n+1}}{n\left(\frac{8}{5}\right)^{n}}\right| = \lim_{n\to\infty}\left|\frac{(n+1)}{n}\left(\frac{8}{5}\right)\right| = 1\cdot\left(\frac{8}{5}\right)$$

Therefore $\sum_{n=1}^{\infty} n\left(\frac{8}{5}\right)^{n}$ diverges by Ratio Test $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| > 1.$

Show that $\sum_{n=1}^{\infty} \frac{n}{7^n}$ converges by Ratio Test.

Let: $a_n = \frac{n}{7^n}$ and $a_{n+1} = \frac{n+1}{7^{n+1}}$

Note: $7^{n+1} = 7^n \cdot 7^1$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\frac{n+1}{7^{n+1}}}{\frac{n}{7^n}}\right| = \lim_{n\to\infty}\left|\frac{n+1}{n}\right| \cdot \lim_{n\to\infty}\left|\frac{7^n}{7^{n+1}}\right|$$

$$\lim_{n\to\infty}\left|\frac{n+1}{n}\right| \cdot \lim_{n\to\infty}\left|\frac{7^n}{7^n \cdot 7}\right| = 1 \cdot \frac{1}{7} = \frac{1}{7}$$

Therefore $\sum_{n=1}^{\infty} \frac{n}{7^n}$ converges by Ratio Test because $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| < 1.$

Show that $\sum_{n=1}^{\infty} \frac{n^4}{5^n}$ converges by Ratio Test.

Let: $a_n = \frac{n^4}{5^n}$ and $a_{n+1} = \frac{(n+1)^4}{5^{n+1}}$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\frac{(n+1)^4}{5^{n+1}}}{\frac{n^4}{5^n}}\right| = \lim_{n\to\infty}\left|\frac{(n+1)^4}{n^4}\right| \cdot \lim_{n\to\infty}\left|\frac{5^n}{5^{n+1}}\right|$$

$$= \lim_{n\to\infty}\left|\frac{(n+1)^4}{n^4}\right| \cdot \lim_{n\to\infty}\left|\frac{5^n}{5^n \cdot 5}\right| = 1 \cdot \frac{1}{5} = \frac{1}{5}$$

Therefore, $\sum_{n=1}^{\infty} \frac{n^4}{5^n}$ converges by Ratio Test because $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| < 1$.

Show that $\sum_{n=1}^{\infty} \frac{(-1)^n 6^n}{(n+1)!}$ converges.

Let: $a_n = \frac{(-1)^n 6^n}{(n+1)!}$ and $a_{n+1} = \frac{(-1)^{n+1} 6^{n+1}}{(n+2)!}$

Note: $6^{n+1} = 6^n \cdot 6^1$ ; $(n+1)! = (n+1)(n)(n-1)\cdots(3)(2)(1);$

$$(n+2)! = (n+2)(n+1)(n)(n-1)\cdots(3)(2)(1)$$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\frac{(-1)^{n+1} 6^{n+1}}{(n+2)!}}{\frac{(-1)^n 6^n}{(n+1)!}}\right| = \lim_{n\to\infty}\left|\frac{(n+1)!}{(n+2)!}\right| \cdot \lim_{n\to\infty}\left|\frac{6^{n+1}}{6^n}\right|$$

$$= \lim_{n\to\infty}\left|\frac{1}{(n+2)}\right| \cdot \lim_{n\to\infty}|6| = \frac{6}{\infty} = 0 < 1$$

Therefore the series $\sum_{\text{n}=1}^{\infty} \frac{(-1)^n 6^n}{(n+1)!}$ converges by Ratio Test.

Show that $\sum_{n=1}^{\infty} \frac{n!}{n \cdot 8^n}$ diverges.

Let: $a_n = \frac{n!}{n \cdot 8^n}$ and $a_{n+1} = \frac{(n+1)!}{(n+1) \cdot 8^{n+1}}$

Note: $(n+1)! = (n+1)(n)(n-1)\cdots(3)(2)(1);$

$$n! = (n)(n-1)\cdots(3)(2)(1)$$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\frac{(n+1)!}{(n+1)\cdot 8^{n+1}}}{\frac{n!}{n\cdot 8^n}}\right| = \lim_{n\to\infty}\left|\frac{(n+1)!}{n!}\right| \cdot \lim_{n\to\infty}\left|\frac{n}{n+1}\right| \cdot \lim_{n\to\infty}\left|\frac{8^n}{8^{n+1}}\right|$$

$$= \lim_{n\to\infty}|n+1| \cdot \lim_{n\to\infty}\left|\frac{n}{n+1}\right| \cdot \lim_{n\to\infty}\left|\frac{1}{8}\right| = \infty(1)\left(\frac{1}{8}\right) = \infty$$

Therefore the series $\sum_{n=1}^{\infty} \frac{n!}{n \cdot 8^n}$ diverges by Ratio Test because $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| > 1$

Show that $\sum_{n=1}^{\infty} \frac{3^n}{5^n+1}$ converges.

Let: $a_n = \frac{3^n}{5^n+1}$ and $a_{n+1} = \frac{3^{n+1}}{5^{n+1}+1}$; Note: $3^{n+1} = 3^n \cdot 3^1$;

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\dfrac{3^{n+1}}{5^{n+1}+1}}{\dfrac{3^n}{5^n+1}}\right| = \lim_{n\to\infty}\left|\frac{3^{n+1}}{5^{n+1}+1}\cdot\frac{5^n+1}{3^n}\right|$$

$$= \lim_{n\to\infty}\left|\frac{3^{n+1}}{3^n}\cdot\frac{5^n+1}{5^{n+1}+1}\right| = \lim_{n\to\infty}\left|3\cdot\frac{5^n+1}{5^{n+1}+1}\right| = 3\lim_{n\to\infty}\left|\frac{5^n+1}{5^n\cdot 5+1}\right|$$

$$= 3\lim_{n\to\infty}\left|\frac{(\ln 5)5^n}{5\cdot(\ln 5)5^n}\right| = 3\lim_{n\to\infty}\left|\frac{1}{5}\right| = \frac{3}{5}$$

Using L'Hopital's Rule; $D_x\left(5^x\right) = \ln 5\cdot 5^x$

Therefore $\sum_{n=1}^{\infty} \frac{3^n}{5^n+1}$ converges by Ratio Test.

Root Test

If $\lim_{x\to\infty} \sqrt[n]{|a_n|} < 1$, then the series $\sum_{n=1}^{\infty} a_n$ converges.

If $\lim_{x\to\infty} \sqrt[n]{|a_n|} > 1$, then the series $\sum_{n=1}^{\infty} a_n$ diverges.

If $\lim_{x\to\infty} \sqrt[n]{|a_n|} = 1$, use another test.

Show that $\sum_{n=1}^{\infty} \frac{1}{8^n}$ converges.

Note: $a_n = \frac{1}{8^n}$

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \sqrt[n]{\left|\frac{1}{8^n}\right|} = 1/8$$

$\sum_{n=1}^{\infty} \frac{1}{8^n}$ converges by Root Test because $\lim_{n\to\infty} \sqrt[n]{|a_n|} < 1.$

Show that $\displaystyle\sum_{n=1}^{\infty}\left(\frac{5n}{2n+1}\right)^{n}$ converges.

Note: $a_n = \left(\dfrac{5n}{2n+1}\right)^{n}$

$$\lim_{n\to\infty}\sqrt[n]{|a_n|} = \lim_{n\to\infty}\sqrt[n]{\left|\left(\frac{5n}{2n+1}\right)^{n}\right|} = \lim_{n\to\infty}\frac{5n}{2n+1} = \frac{5}{2}$$

Therefore $\displaystyle\sum_{n=1}^{\infty}\left(\frac{5n}{2n+1}\right)^{n}$ converges by Root Test because $\displaystyle\lim_{n\to\infty}\sqrt[n]{|a_n|} > 1$.

Show that $\sum_{n=1}^{\infty}\left(5\sqrt[n]{n}+2\right)^{n}$ diverges.

Note: $a_n = \left(5\sqrt[n]{n}+2\right)^{n}$

Hint: $\lim_{n\to\infty} \sqrt[n]{n} = \lim_{n\to\infty} n^{1/n} = 1.$

$$\lim_{n\to\infty}\sqrt[n]{\left|a^n\right|} = \lim_{n\to\infty}\sqrt[n]{\left|\left(5\sqrt[n]{n}+2\right)^{n}\right|} = \lim_{n\to\infty}\left(5\sqrt[n]{n}+2\right)$$

$$= \lim_{n\to\infty}\left(5n^{1/n}+2\right) = 5\left(1\right)+2 = 7$$

Therefore, $\sum_{n=1}^{\infty}\left(5\sqrt[n]{n}+2\right)^{n}$ diverges by Root Test because $\lim_{n\to\infty}\sqrt[n]{\left|a^n\right|} > 1.$

Show that $\sum_{n=1}^{\infty} \frac{2n}{5^n}$ converges.

Note: $a_n = \frac{2n}{5^n}$.

Hint: $\lim_{n\to\infty} \sqrt[n]{n} = \lim_{n\to\infty} n^{1/n} = 1$; $\quad \lim_{n\to\infty} \text{constant}^{1/n} = 1$

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \sqrt[n]{\left|\frac{2n}{5^n}\right|} = \lim_{n\to\infty} \frac{\sqrt[n]{2n}}{\sqrt[n]{5^n}} = \lim_{n\to\infty} \frac{\sqrt[n]{2}\sqrt[n]{n}}{5}$$

$$= \lim_{n\to\infty} \frac{2^{1/n} \cdot n^{1/n}}{5} = \lim_{n\to\infty} \frac{1 \cdot 1}{5} = \frac{1}{5}$$

Therefore, $\sum_{n=1}^{\infty} \frac{2n}{5^n}$ converges by Root Test because $\lim_{n\to\infty} \sqrt[n]{|a_n|} < 1$.

Show that $\sum_{n=1}^{\infty} \frac{5n}{(\ln n)^n}$ converges.

Note: $a_n = \frac{5n}{(\ln n)^n}$

Hint: $\lim_{n\to\infty} \sqrt[n]{n} = \lim_{n\to\infty} n^{1/n} = 1$;    $\lim_{n\to\infty} \text{constant}^{1/n} = 1$

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \sqrt[n]{\left|\frac{5n}{(\ln n)^n}\right|} = \lim_{n\to\infty} \frac{\sqrt[n]{5n}}{\sqrt[n]{(\ln n)^n}} = \lim_{n\to\infty} \frac{5^{1/n} \cdot n^{1/n}}{\ln(n)}$$

$$= \frac{1\cdot 1}{\ln(\infty)} = \frac{1}{\infty} = 0$$

Therefore $\sum_{n=1}^{\infty} \frac{5n}{(\ln n)^n}$ converges by Root Test because $\lim_{n\to\infty} \sqrt[n]{|a_n|} < 1$.