n*th* Degree Taylor Polynomial:

$$P_n(x) = f(c) + f'(c)(x-c) + \frac{f''(c)(x-c)^2}{2} + \frac{f'''(c)(x-c)^3}{3!}$$

$$+\frac{f^{(4)}(c)(x-c)^4}{4!} + \cdots + \frac{f^{(n)}(c)(x-c)^n}{n!}$$

Example 1: Let $f(x) = \sin x$

Find a second-degree Taylor polynomial $P_2(x)$ centered at $c = \dfrac{\pi}{4}$

$$f(x) = \sin x \quad \Rightarrow \quad f\left(\frac{\pi}{4}\right) = \sin(\pi/4) = \sqrt{2}/2$$

$$f'(x) = \cos x \quad \Rightarrow \quad f'\left(\frac{\pi}{4}\right) = \cos(\pi/4) = \sqrt{2}/2$$

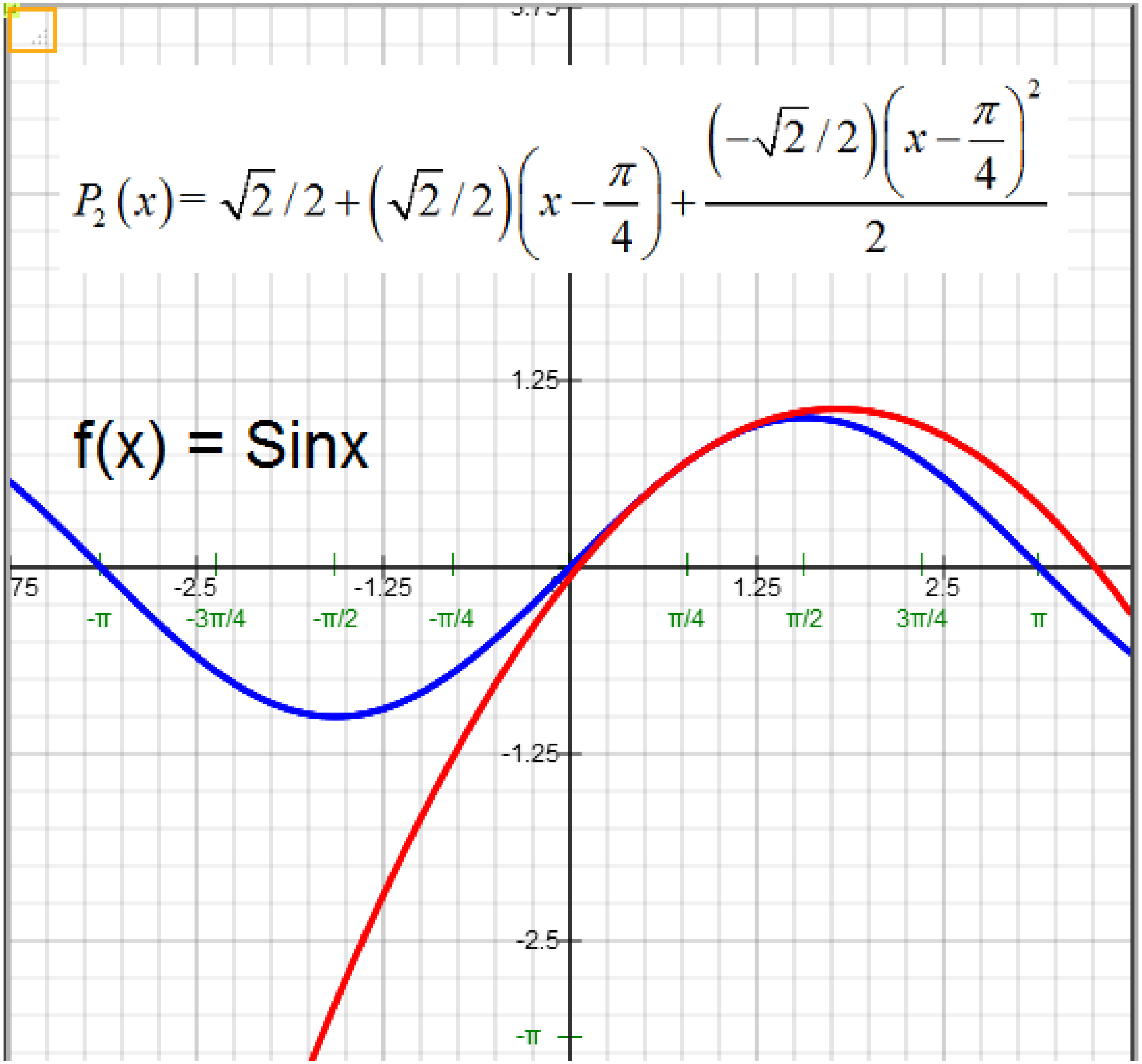
$$f''(x) = -\sin x \quad \Rightarrow \quad f''\left(\frac{\pi}{4}\right) = -\sin(\pi/4) = -\sqrt{2}/2$$

$$P_2(x) = f(c) + f'(c)(x-c) + \frac{f''(c)(x-c)^2}{2}$$

$$P_2(x) = f(\pi/4) + f'(\pi/4)(x-\pi/4) + \frac{f''(\pi/4)(x-\pi/4)^2}{2}$$

$$P_2(x) = \sqrt{2}/2 + \left(\sqrt{2}/2\right)(x-\pi/4) + \frac{\left(-\sqrt{2}/2\right)(x-\pi/4)^2}{2}$$

Comparing $f(x)$ and $P_2(x)$



Example 2: Let $f(x)=\cot x$

Find a second-degree polynomial $P_2(x)$ centered at $c=\dfrac{\pi}{3}$

$$f(x)=\cot x \quad \Rightarrow \quad f(\pi/3)= \cot x = \cot(\pi/3) = 1/\sqrt{3}$$

$$f'(x)= -(\csc x)^2 \quad \Rightarrow \quad f'(\pi/3)= -(\csc x)^2 = -(\csc(\pi/3))^2 = -(2/\sqrt{3})^2 = -4/3$$

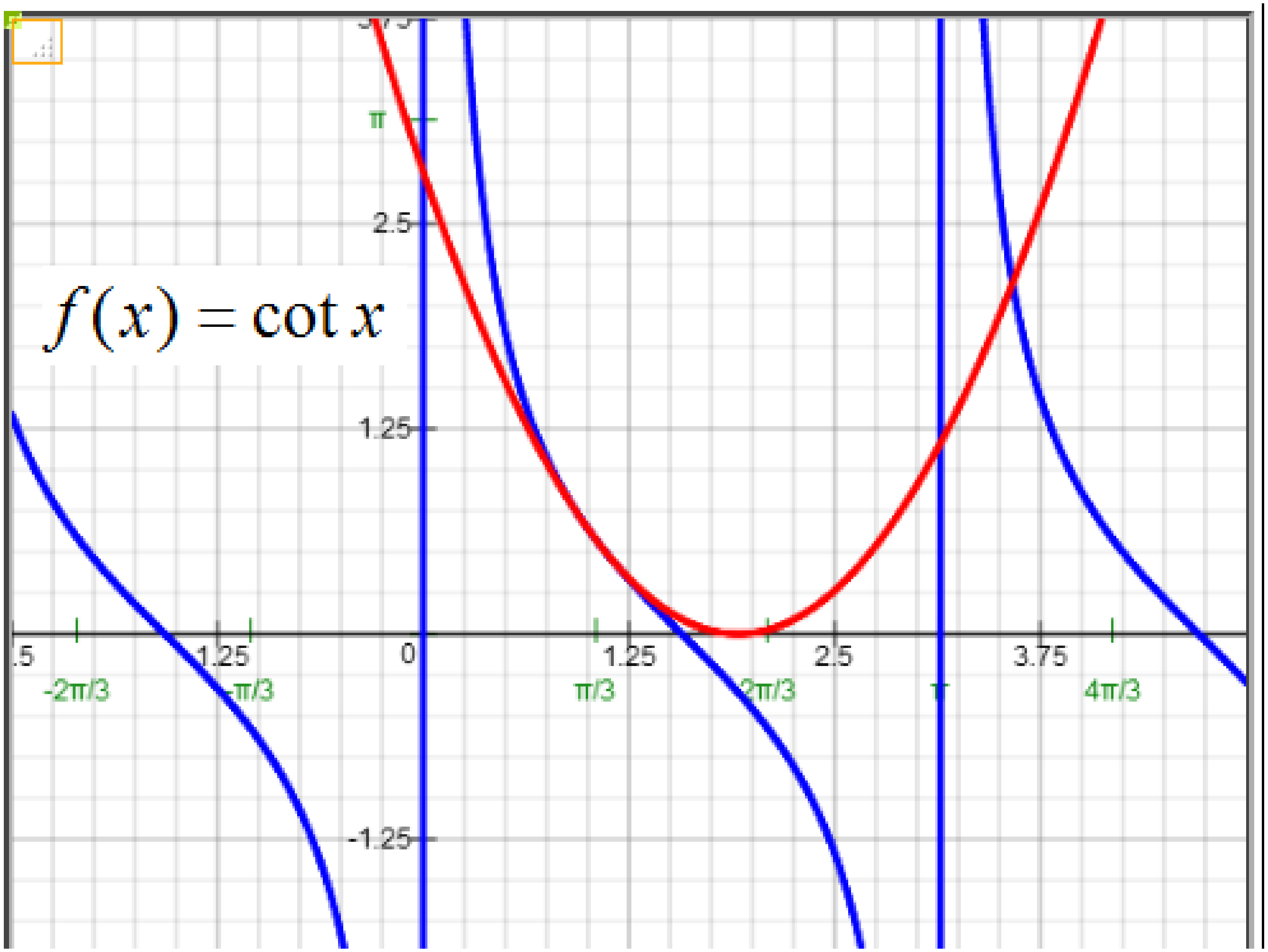
$$f''(x)= -2(\csc x)(-\csc x\cdot\cot x)= 2(\csc x)(\csc x\cdot\cot x) \quad \Rightarrow \quad f''(\pi/3)= 8/\left(\sqrt{3}\right)^3$$

$$P_2(x)= f(c)+ f'(c)(x-c)+\frac{f''(c)(x-c)^2}{2}$$

$$P_2(x)= f(\pi/3)+ f'(\pi/3)(x-\pi/3)+\frac{f''(\pi/3)(x-\pi/3)^2}{2}$$

$$P_2(x)=1/\sqrt{3}+(-4/3)(x-\pi/3)+\frac{8/\left(\sqrt{3}\right)^3(x-\pi/3)^2}{2}$$

Comparing $f(x)$ and $P_2(x)$



Example 3: Let $f(x) = \dfrac{4}{\sqrt{x^3}} = 4x^{-3/2}$

Find a second-degree polynomial $P_2(x)$ centered at $c = 1$.

$$f(x) = \frac{4}{\sqrt{x^3}} = 4x^{-3/2} \qquad f(1) = \frac{4}{\sqrt{x^3}} = \frac{4}{\sqrt{1^3}} = 4;$$

$$f'(x) = 4\left(\frac{-3}{2}x^{-5/2}\right) = -6x^{-5/2}; \qquad f'(1) = -6x^{-5/2} = -6(1)^{-5/2} = -6$$

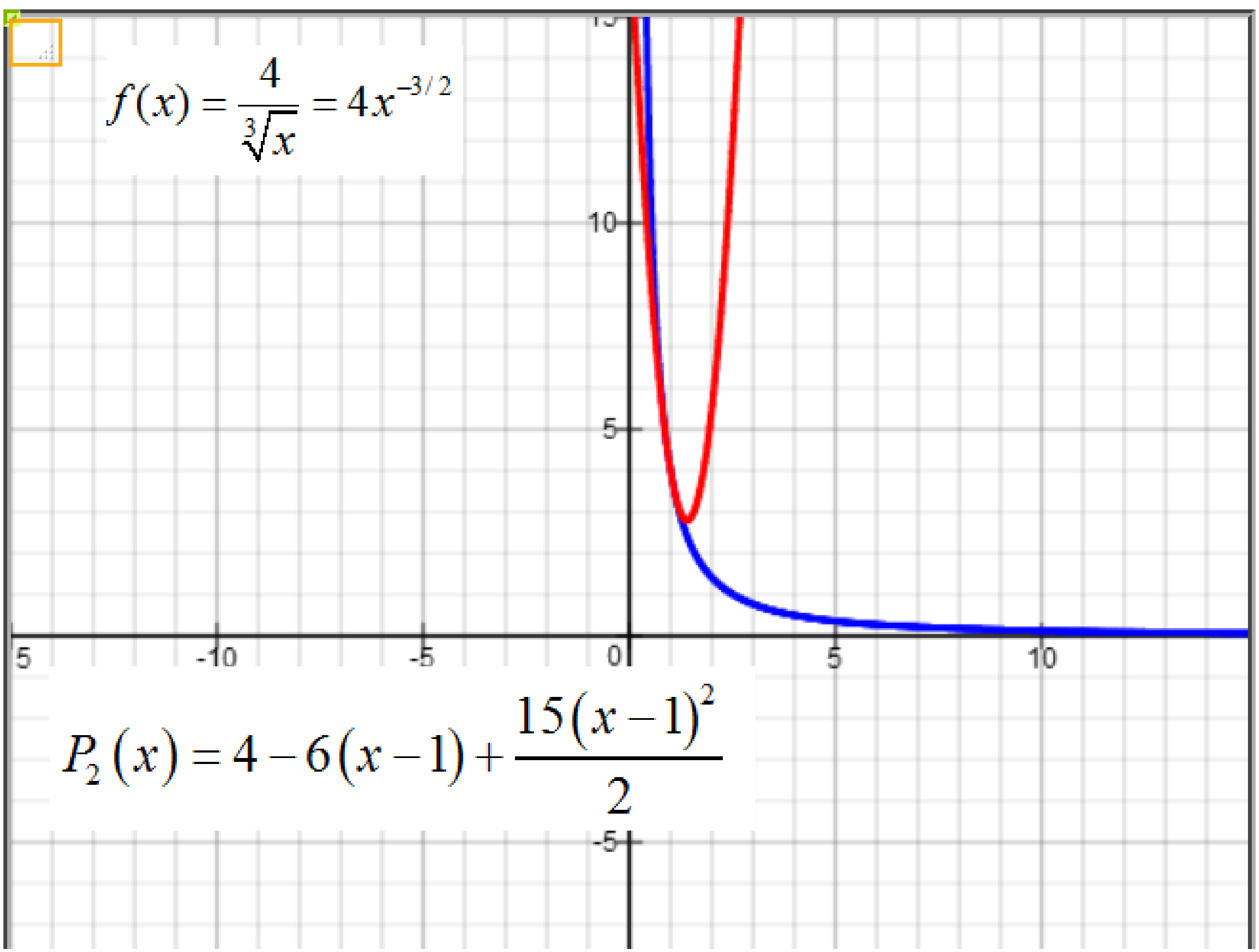
$$f''(x) = -6\left(\frac{-5}{2}x^{-7/2}\right) = 15x^{-7/2} \qquad f''(1) = 15x^{-7/2} = 15(1)^{-7/2} = 15$$

$$P_2(x) = f(c) + f'(c)(x-c) + \frac{f''(c)(x-c)^2}{2}$$

$$P_2(x) = f(1) + f'(1)(x-1) + \frac{f''(1)(x-1)^2}{2}$$

$$P_2(x) = 4 - 6(x-1) + \frac{15(x-1)^2}{2}$$

Comparing $f(x)$ and $P_2(x)$



Example 4: Let $f(x)=5xe^{x}$

Find a second-degree polynomial $P_2(x)$ centered at $c=0$.

$$f(x)=5xe^{x}$$

$$f'(x)=(5x)D_x(e^{x})+(e^{x})D_x(5x)=5xe^{x}+e^{x}(5)=5xe^{x}+5e^{x}$$

$$f''(x)=5xe^{x}+5e^{x}+5e^{x}$$

$$f(0)=5xe^{x}=5(0)e^{0}=0(1)=0$$

$$f'(0)=5xe^{x}+5e^{x}=0+5=5$$

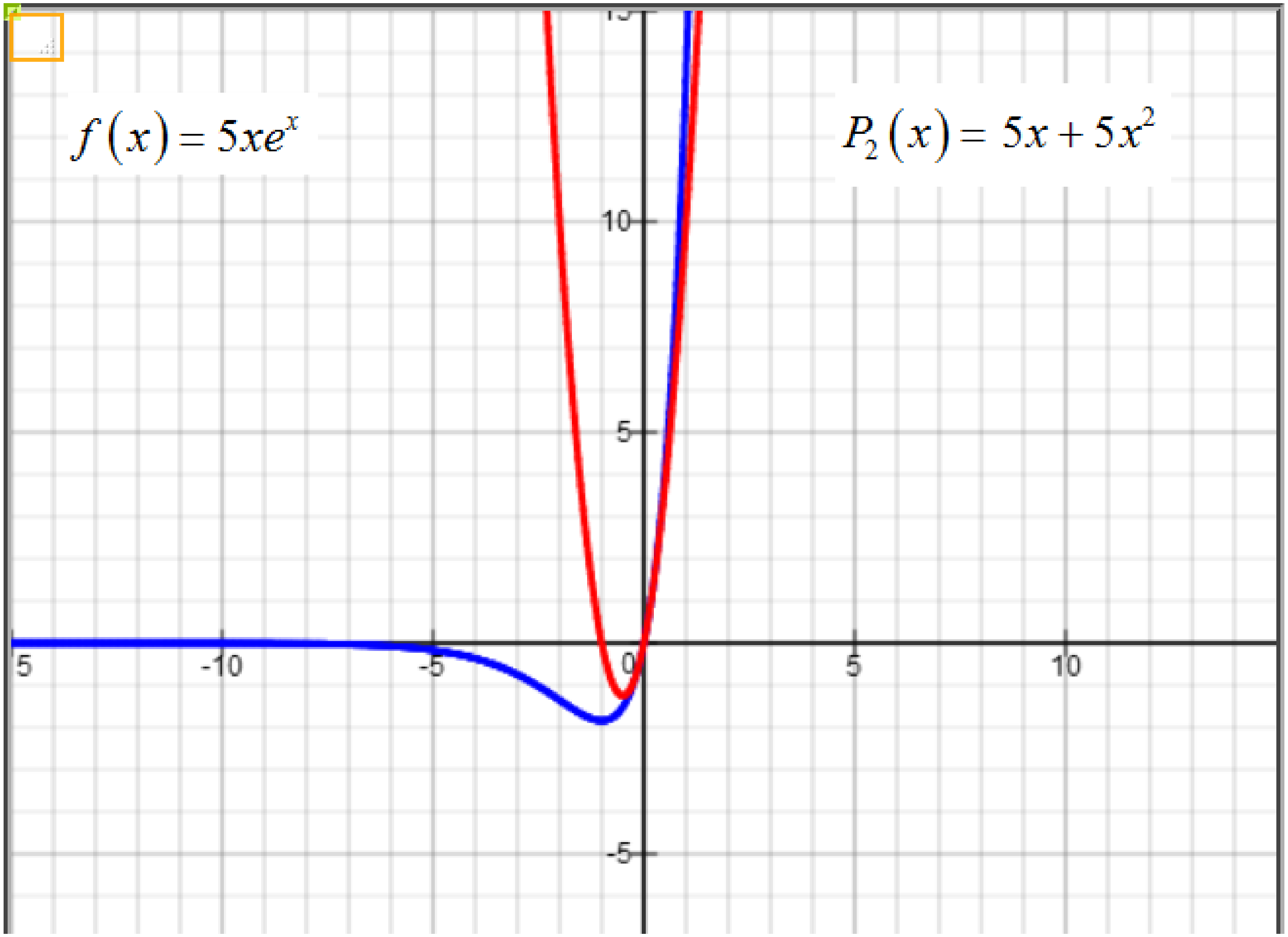
$$f''(0)=5xe^{x}+10e^{x}=0+10=10$$

$$P_2(x)=f(c)+f'(c)(x-c)+\frac{f''(c)(x-c)^2}{2}$$

$$P_2(x)=f(0)+f'(0)(x-0)+\frac{f''(0)(x-0)^2}{2}$$

$$P_2(x)=0+5(x)+\frac{10(x)^2}{2}=5x+5x^2$$

Comparing $f(x)$ and $P_2(x)$



Example 5: Let $f(x)=\dfrac{4}{x+2}=4(x+2)^{-1}$

Find a second-degree polynomial $P_2(x)$ centered at $c=1$.

$$f(x)=\frac{4}{x+2}=4(x+2)^{-1}$$

$$f'(x)=4\left[-1(x+2)^{-2}\cdot D(x-2)\right]=4\left[-1(x+2)^{-2}\cdot(1)\right]=-4(x+2)^{-2}$$

$$f''(x)=-4\left[-2(x+2)^{-3}\cdot D(x-2)\right]=-4\left[-2(x+2)^{-3}\cdot(1)\right]=8(x+2)^{-3}$$

$$f(1)=\frac{4}{x+2}=\frac{4}{1+2}=4/3$$

$$f'(1)=-4(x+2)^{-2}=-4(1+2)^{-2}=-4(3)^{-2}=-4\left(\frac{1}{9}\right)=\frac{-4}{9}$$

$$f''(1)=8(x+2)^{-3}=8(1+2)^{-3}=8(3)^{-3}=8\left(\frac{1}{27}\right)=\frac{8}{27}$$

$$P_2(x)=f(c)+f'(c)(x-c)+\frac{f''(c)(x-c)^2}{2}$$

$$P_2(x)=f(1)+f'(1)(x-1)+\frac{f''(1)(x-1)^2}{2}$$

$$P_2(x)=4/3-(4/9)(x-1)+\frac{(8/27)(x-1)^2}{2}$$

Comparing $f(x)$ and $P_2(x)$