Calculus II Section 9.8 Notes

Ratio Test:

If $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|<1$, then $\sum_{n=1}^{\infty}a_n$ converges.

If $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|>1$, then $\sum_{n=1}^{\infty}a_n$ diverges.

If $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=1$, then use another test.

Example 1:

$\sum_{n=0}^{\infty}(-1)^n \frac{x^n}{2n+8}$ is called a power series.

Find all values for x that would make the series converge.

Let $u_n = (-1)^n \frac{x^n}{2n+8}$; $\quad u_{n+1} = (-1)^{n+1} \frac{x^{n+1}}{2(n+1)+8} = (-1)^{n+1} \frac{x^{n+1}}{2n+10}$

Let $L = \lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{(-1)^{n+1}\frac{x^{n+1}}{2n+10}}{(-1)^n \frac{x^n}{2n+8}}\right| = \lim_{n\to\infty}\left|\frac{x^n \cdot x^1}{x^n}\right| \cdot \lim_{n\to\infty}\left|\frac{2n+8}{2n+10}\right| = |x|(1) = |x|$

By Ratio Test, if $L<1$ then $\sum_{n=0}^{\infty}(-1)^n \frac{x^n}{2n+8}$ converges.

Hence, $L<1 \quad \Leftrightarrow \quad |x|<1 \quad \Leftrightarrow \quad -1<x<1$

R = Radius of convergence = 1

Interval of Convergence = $(-1, 1)$.

Hence the series $\sum_{n=0}^{\infty}(-1)^n \frac{x^n}{2n+8}$ converges when x is between -1 and 1.

Example 2:

$\sum_{n=1}^{\infty}\frac{(8x)^n}{n^3}$ is called a power series. Find radius of convergence.

Let $u_n = \frac{(8x)^n}{n^3}$ and $u_{n+1} = \frac{(8x)^{n+1}}{\left(n+1\right)^3}$

Let $L = \lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{(8x)^{n+1}}{\left(n+1\right)^3}\cdot\frac{n^3}{(8x)^n}\right| = \lim_{n\to\infty}\left|\frac{(8x)^{n+1}}{(8x)^n}\right|\cdot\lim_{n\to\infty}\left|\frac{n^3}{(n+1)^3}\right|$

$$= \lim_{n\to\infty}\left|\frac{(8x)^n\cdot\left(8x\right)^1}{(8x)^n}\right|\cdot\lim_{n\to\infty}\left|\frac{n^3}{n^3+3n+3n+1}\right| = |8x|\cdot\left(1\right) = |8x|$$

By Ratio Test, if $L<1$ then $\sum_{n=0}^{\infty}\frac{(8x)^n}{n^3}$ converges.

Hence, $L<1 \Leftrightarrow \ |8x|<1 \ \Leftrightarrow \ -1<8x<1 \ \Rightarrow \ -1/8<x<1/8$

We also need to check $x=-1/8$ and $x=1/8$.

Example 2 $(con't)$.

If $x = -1/8$, $\sum_{n=1}^{\infty} \frac{(8x)^n}{n^3} = \sum_{n=1}^{\infty} \frac{(8(-1/8))^n}{n^3} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^3}$

$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^3}$ is an alternating series and converges by Alternating Series Test.

If $x = 1/8$, $\sum_{n=1}^{\infty} \frac{(8x)^n}{n^3} = \sum_{n=1}^{\infty} \frac{(8(1/8))^n}{n^3} = \sum_{n=1}^{\infty} \frac{(1)^n}{n^3} = \sum_{n=1}^{\infty} \frac{1}{n^3}$

$\sum_{n=1}^{\infty} \frac{1}{n^3}$ is a $p-$series with $p = 3$; hence it converges.

Summary: Interval of convergence is [-1/8, 1/8]

In other words, $\sum_{n=1}^{\infty} \frac{(8x)^n}{n^3}$ converges when $x \in$ [-1/8, 1/8].

Example 3:

Find interval of convergence for $\sum_{n=1}^{\infty} \frac{x^n}{(n+2)!}$.

Let $u_n = \frac{x^n}{(n+2)!}$ and $u_{n+1} = \frac{x^{n+1}}{(n+1+2)!} = \frac{x^{n+1}}{(n+3)!}$

Note: $(n+2)! = (n+2)(n+1)(n)\cdots(3)(2)(1)$

$(n+3)! = (n+3)(n+2)(n+1)(n)\cdots(3)(2)(1)$

$$\text{Let } L = \lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{\frac{x^{n+1}}{(n+3)!}}{\frac{x^n}{(n+2)!}}\right| = \lim_{n\to\infty}\left|\frac{x^{n+1}}{(n+3)!}\cdot\frac{(n+2)!}{x^n}\right|$$

$$= \lim_{n\to\infty}\left|\frac{x^{n+1}}{x^n}\right| \lim_{n\to\infty}\left|\frac{(n+2)!}{(n+3)!}\right| = \lim_{n\to\infty}\left|\frac{x^n x^1}{x^n}\right| \lim_{n\to\infty}\left|\frac{(n+2)!}{(n+3)!}\right|$$

$$= |x| \lim_{n\to\infty}\left|\frac{1}{n+3}\right| = |x|\cdot 0 = 0$$

By Ratio Test, if $L<1$ then $\sum_{n=1}^{\infty} \frac{x^n}{(n+2)!}$ converges.

Note that $L = \lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = 0 < 1$ for any x.

Example 3 (con't):

Hence $\sum_{n=1}^{\infty} \frac{x^n}{(n+2)!}$ converges for any *x*.

Interval of convergence = $\left(-\infty,\infty\right)$

R = Radius of convergence = ∞

Example 4:

Find Interval of convergence for $\sum_{n=1}^{\infty}\left(\frac{x}{5}\right)^n$

Let $u_n = \left(\frac{x}{5}\right)^n$ and $u_{n+1} = \left(\frac{x}{5}\right)^{n+1}$

Hint: $\left(\frac{x}{5}\right)^{n+1} = \left(\frac{x}{3}\right)^n \left(\frac{x}{3}\right)^1$

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{\left(\frac{x}{5}\right)^{n+1}}{\left(\frac{x}{5}\right)^n}\right| = \lim_{n\to\infty}\left|\frac{\left(\frac{x}{3}\right)^n\left(\frac{x}{3}\right)^1}{\left(\frac{x}{5}\right)^n}\right| = \left|\frac{x}{5}\right|$$

By Ratio Test, if $L<1$ then $\sum_{n=1}^{\infty}\left(\frac{x}{5}\right)^n$ converges.

$$L<1 \quad \Leftrightarrow \quad \left|\frac{x}{5}\right|<1 \quad \Rightarrow \quad -1<\frac{x}{5}<1 \quad \Rightarrow \quad -5<x<5$$

Example 4 (con't):

We also need to test $x = 5$ and $x = -5$.

For $x = 5, \sum_{n=1}^{\infty}\left(\frac{x}{5}\right)^n = \sum_{n=1}^{\infty}\left(\frac{5}{5}\right)^n = \sum_{n=1}^{\infty}(1)^n = \infty.$

Hence, when $x = 5, \sum_{n=1}^{\infty}\left(\frac{x}{5}\right)^n$ diverges.

For $x = -5, \sum_{n=1}^{\infty}\left(\frac{x}{5}\right)^n = \sum_{n=1}^{\infty}\left(\frac{-5}{5}\right)^n = \sum_{n=1}^{\infty}(-1)^n = -1+1-1+1-1+\ldots$

Hence, when $x = -5, \sum_{n=1}^{\infty}\left(\frac{x}{5}\right)^n$ diverges.

Therefore, Interval of Convergence is $(-5, 5)$.

Example 5:

Find Interval of convergence for $\sum_{n=1}^{\infty} \frac{x^{3n}}{(n+1)!}$.

Let $u_n = \frac{x^{3n}}{(n+1)!}$ and $u_{n+1} = \frac{x^{3(n+1)}}{((n+1)+1)!} = \frac{x^{3n+3}}{(n+2)!}$

Note: $(n+1)! = (n+1)(n)(n-1)(n-2)\cdots(3)(2)(1)$

$(n+2)! = (n+2)(n+1)(n)(n-1)(n-2)(n-3)\cdots(3)(2)(1)$

Let $L = \lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{x^{3n+3}}{(n+2)!}\,\frac{(n+1)!}{x^{3n}}\right| = \lim_{n\to\infty}\left|\frac{x^{3n}\cdot x^3}{x^{3n}}\right| \cdot \lim_{n\to\infty}\left|\frac{(n+1)!}{(n+2)!}\right|$

$= \left|x^3\right| \cdot \lim_{n\to\infty}\left|\frac{1}{(n+2)}\right| = \left|x^3\right| \cdot 0 = 0$

By Ratio Test, if $L<1$ then $\sum_{n=1}^{\infty} \frac{x^{3n}}{(n+1)!}$ converges.

Since $L=0$, $L<1$ for any x. Hence, $\sum_{n=1}^{\infty} \frac{x^{3n}}{(n+1)!}$ converges for any x.

Therefore Interval of convergence $= (-\infty, \infty)$ and Radius of Convergence $= \infty$

Example 6:

Find Interval of convergence for $\sum_{n=1}^{\infty}\frac{(x-2)^{n-1}}{2^{n-1}}$.

Let $u_n=\frac{(x-2)^{n-1}}{2^{n-1}}$ and $u_{n+1}=\frac{(x-2)^{(n+1)-1}}{2^{(n+1)-1}}=\frac{(x-2)^n}{2^n}$

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right|=\lim_{n\to\infty}\left|\frac{(x-2)^n}{2^n}\cdot\frac{2^{n-1}}{(x-2)^{n-1}}\right|$$

$$=\lim_{n\to\infty}\left|\frac{(x-2)^n}{(x-2)^n\cdot(x-2)^{-1}}\right|\cdot\lim_{n\to\infty}\left|\frac{2^n\cdot 2^{-1}}{2^n}\right|$$

$$=\lim_{n\to\infty}\left|\frac{1}{(x-2)^{-1}}\right|\cdot\lim_{n\to\infty}\left|2^{-1}\right|=\lim_{n\to\infty}\left|(x-2)^1\right|\cdot\lim_{n\to\infty}\left|\frac{1}{2}\right|=\frac{1}{2}|x-2|$$

By Ratio Test, if $L<1$ then $\sum_{n=1}^{\infty}\frac{(x-2)^{n-1}}{2^{n-1}}$ converges.

Hence, $L<1 \quad\Leftrightarrow\quad \frac{1}{2}|x-2|<1 \quad\Leftrightarrow\quad -1<\frac{1}{2}\left(x-2\right)<1$

$$\Leftrightarrow\quad -2<\left(x-2\right)<2 \quad\Leftrightarrow\quad 0<x<4$$

Example 6 (con't):

We also to need to test $x = 0$ and $x = 4$.

For $x = 0, \sum_{n=1}^{\infty} \frac{(x-2)^{n-1}}{2^{n-1}} = \sum_{n=1}^{\infty} \frac{(0-2)^{n-1}}{2^{n-1}} = \sum_{n=1}^{\infty} \frac{(-2)^{n-1}}{2^{n-1}}$

$$= \sum_{n=1}^{\infty} \frac{(-1 \cdot 2)^{n-1}}{2^{n-1}} = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}(2)^{n-1}}{2^{n-1}} = \sum_{n=1}^{\infty} (-1)^{n-1}$$

For $\sum_{n=1}^{\infty} (-1)^{n-1}$, note that $a_n = \left|(-1)^{n-1}\right| = 1^n = 1;$ and $\lim_{n\to\infty} a_n = \lim_{n\to\infty} 1 = 1$

Hence, $\sum_{n=1}^{\infty} (-1)^{n-1}$ diverges by n^{th} Term Test.

Therefore, for $x = 0, \sum_{n=1}^{\infty} \frac{(x-2)^{n-1}}{2^{n-1}}$ diverges.

Example 6 (con't):

For $x = 4, \sum_{n=1}^{\infty} \frac{(4-2)^{n-1}}{2^{n-1}} = \sum_{n=1}^{\infty} \frac{(2)^{n-1}}{2^{n-1}} = \sum_{n=1}^{\infty} 1 = 1+1+1+\ldots = \infty$

Hence, For $x = 4, \sum_{\text{n}=1}^{\infty} \frac{(x-2)^{n-1}}{2^{n-1}}$ diverges.

Therefore, Interval of convergence = (0, 4).

Radius of Convergence = 2.

Example 7:

Find the interval of convergence for $\sum_{n=1}^{\infty} \frac{x^{5n+1}}{(5n+1)!}$.

Let $u_n = \frac{x^{5n+1}}{(5n+1)!}$ and $u_{n+1} = \frac{x^{5(n+1)+1}}{(5(n+1)+1)!} = \frac{x^{5n+6}}{(5n+6)!}$

Note: $(5n+1)! = (5n+1)(5n)(5n-1)(5n-2)(5n-3)\cdots(3)(2)(1)$

$(5n+6)! = (5n+6)(5n+5)(5n+4)(5n+2)(5n+1)\cdots(3)(2)(1)$

Let $L = \lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{x^{5n+6}}{(5n+6)!}\,\frac{(5n+1)!}{x^{5n+1}}\right| = \lim_{n\to\infty}\left|\frac{x^{5n}x^{6}}{x^{5n}x^{1}}\right|\cdot\lim_{n\to\infty}\left|\frac{(5n+1)!}{(5n+6)!}\right|$

$$= \lim_{n\to\infty}\left|x^5\right|\cdot\lim_{n\to\infty}\left|\frac{1}{(5n+6)(5n+5)(5n+4)(5n+2)}\right| = \left|x^5\right|\cdot 0 = 0$$

By Ratio Test, if $L<1$ then $\sum_{n=1}^{\infty} \frac{x^{5n+1}}{(5n+1)!}$ converges.

Hence, $L = \lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = 0 < 1$ for any x; and $\sum_{n=1}^{\infty} \frac{x^{5n+1}}{(5n+1)!}$ converges for any real number x.

Therefore, Interval of Convergence $(-\infty,\infty)$ and Radius of Convergence = ∞.

Example 8:

Find the interval of convergence for $\sum_{n=1}^{\infty}(n+1)!x^n$.

Let $u_n = (n+1)!x^n$ and $u_{n+1} = (n+1+1)!x^{n+1} = (n+2)!x^n x^1$

Note: $(n+1)! = (n+1)(n)(n-1)(n-2)(n-3)\cdots(3)(2)(1)$

$(n+2)! = (n+2)(n+1)(n)(n-1)(n-2)(n-3)\cdots(3)(2)(1)$

$$L = \lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{(n+2)!x^n x^1}{(n+1)!x^n}\right| = \lim_{n\to\infty}\left|\frac{(n+2)!}{(n+1)!}\right||x|$$

$$= \lim_{n\to\infty}\left|\frac{(n+2)(n+1)(n)(n-1)(n-2)(n-3)\cdots(3)(2)(1)}{(n+1)(n)(n-1)(n-2)(n-3)\cdots(3)(2)(1)}\right||x|$$

$$L = \lim_{n\to\infty}\left|\frac{n+2}{1}\right||x| = \infty\cdot|x| = \infty$$

Example 8 (con't):

Note: $L = \infty > 1$. Hence, the series $\sum_{n=1}^{\infty} (n+1)! x^n$ diverges for any real number x, except $x = 0$.

When $x = 0$, $\sum_{n=1}^{\infty} (n+1)! x^n = \sum_{n=1}^{\infty} (n+1)! \cdot 0^n = \sum_{n=1}^{\infty} (n+1)! \cdot 0 = \sum_{n=1}^{\infty} 0 = 0$.

Therefore, Interval of Convergence is $\{0\}$.