Ratio Test:

If $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|<1$, then $\sum_{n=1}^{\infty} a_n$ converges.

If $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|>1$, then $\sum_{n=1}^{\infty} a_n$ diverges.

If $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=1$, then use another test.

$$\sum_{n=0}^{\infty}(-1)^n \frac{x^n}{2n+8} \text{ is called a power series.}$$

Find all values for x that would make the series converge.

$$\text{Let } u_n = (-1)^n \frac{x^n}{2n+8}; \quad u_{n+1} = (-1)^{n+1} \frac{x^{n+1}}{2(n+1)+8} = (-1)^{n+1} \frac{x^{n+1}}{2n+10}$$

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{(-1)^{n+1} \dfrac{x^{n+1}}{2n+10}}{(-1)^n \dfrac{x^n}{2n+8}}\right| = \lim_{n\to\infty}\left|\frac{x^n \cdot x^1}{x^n}\right| \cdot \lim_{n\to\infty}\left|\frac{2n+8}{2n+10}\right| = |x|(1)$$

$$\text{Set } |x|(1) < 1 \quad \Rightarrow \quad -1 < x < 1$$

R = Radius of convergence = 1

Interval of Convergence = $\left(-1,\ 1\right)$.

Hence the series $\sum_{n=0}^{\infty}(-1)^n \frac{x^n}{2n+8}$ converges when x is between -1 and 1.

$\sum_{n=1}^{\infty} \frac{(8x)^n}{n^3}$ is called a power series. Find radius of convergence.

Let $u_n = \frac{(8x)^n}{n^3}$ and $u_{n+1} = \frac{(8x)^{n+1}}{\left(n+1\right)^3}$

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{(8x)^{n+1}}{\left(n+1\right)^3}\cdot\frac{n^3}{(8x)^n}\right| = \lim_{n\to\infty}\left|\frac{(8x)^{n+1}}{(8x)^n}\right|\cdot\lim_{n\to\infty}\left|\frac{n^3}{(n+1)^3}\right|$$

$$= \lim_{n\to\infty}\left|\frac{(8x)^n\cdot\left(8x\right)^1}{(8x)^n}\right|\cdot\lim_{n\to\infty}\left|\frac{n^3}{n^3+3n+3n+1}\right| = \left|8x\right|\cdot\left(1\right)$$

Set $\left|8x\right| < 1 \;\Rightarrow\; -1 < 8x < 1 \;\Rightarrow\; -1/8 < x < 1/8$

$\sum_{n=1}^{\infty} \frac{(8x)^n}{n^3}$ is called a power series. $(con't)$.

If $x = -1/8$, $\sum_{n=1}^{\infty} \frac{(8x)^n}{n^3} = \sum_{n=1}^{\infty} \frac{(8(-1/8))^n}{n^3} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^3}$

$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^3}$ is an alternating series and converges by Alternating Series Test.

If $x = 1/8$, $\sum_{n=1}^{\infty} \frac{(8x)^n}{n^3} = \sum_{n=1}^{\infty} \frac{(8(1/8))^n}{n^3} = \sum_{n=1}^{\infty} \frac{(1)^n}{n^3} = \sum_{n=1}^{\infty} \frac{1}{n^3}$

$\sum_{n=1}^{\infty} \frac{1}{n^3}$ is a $p-$series with $p = 3;$ hence it converges.

Summary: Interval of convergence is [-1/8, 1/8]

In other words, $\sum_{n=1}^{\infty} \frac{(8x)^n}{n^3}$ converges when $x \in$ [-1/8, 1/8].

Find interval of convergence for $\sum_{\mathrm{n=1}}^{\infty} \frac{x^n}{(n+2)!}$.

Let $u_n = \frac{x^n}{(n+2)!}$ and $u_{n+1} = \frac{x^{n+1}}{(n+1+2)!} = \frac{x^{n+1}}{(n+3)!}$

Note: $(n+2)! = (n+2)(n+1)(n)\cdots(3)(2)(1)$

$(n+3)! = (n+3)(n+2)(n+1)(n)\cdots(3)(2)(1)$

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{\frac{x^{n+1}}{(n+3)!}}{\frac{x^n}{(n+2)!}}\right| = \lim_{n\to\infty}\left|\frac{x^{n+1}}{(n+3)!}\cdot\frac{(n+2)!}{x^n}\right|$$

$$= \lim_{n\to\infty}\left|\frac{x^{n+1}}{x^n}\right| \lim_{n\to\infty}\left|\frac{(n+2)!}{(n+3)!}\right| = \lim_{n\to\infty}\left|\frac{x^n x^1}{x^n}\right| \lim_{n\to\infty}\left|\frac{(n+2)!}{(n+3)!}\right|$$

$$= |x| \lim_{n\to\infty}\left|\frac{1}{n+3}\right| = |x|\cdot 0 = 0$$

Note: $\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| < 1$ for any x.

Find interval of convergence for $\sum_{n=1}^{\infty} \frac{x^n}{(n+2)!}$. con't

Hence $\sum_{n=1}^{\infty} \frac{x^n}{(n+2)!}$ converges for any *x*.

Interval of convergence = $(-\infty, \infty)$

InR = Radius of convergence = ∞

Find Interval of convergence for $\sum_{n=1}^{\infty}\left(\frac{x}{5}\right)^{n}$

Let $u_n = \left(\frac{x}{5}\right)^{n}$ and $u_{n+1} = \left(\frac{x}{5}\right)^{n+1}$

Hint: $\left(\frac{x}{5}\right)^{n+1} = \left(\frac{x}{3}\right)^{n}\left(\frac{x}{3}\right)^{1}$

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{\left(\frac{x}{5}\right)^{n+1}}{\left(\frac{x}{5}\right)^{n}}\right| = \lim_{n\to\infty}\left|\frac{\left(\frac{x}{3}\right)^{n}\left(\frac{x}{3}\right)^{1}}{\left(\frac{x}{5}\right)^{n}}\right| = \frac{x}{3}$$

Set $\frac{x}{5} < 1 \quad \Rightarrow \quad -1 < \frac{x}{5} < 1 \quad \Rightarrow \quad -5 < x < 5$

Find Interval of convergence for $\sum_{n=1}^{\infty}\left(\frac{x}{5}\right)^n$ con't

We also need to test $x = 5$ and $x = -5$.

For $x = 5, \sum_{n=1}^{\infty}\left(\frac{x}{5}\right)^n = \sum_{n=1}^{\infty}\left(\frac{5}{5}\right)^n = \sum_{n=1}^{\infty}(1)^n = \infty.$

Hence, when $x = 5, \sum_{n=1}^{\infty}\left(\frac{x}{5}\right)^n$ diverges.

For $x = -5, \sum_{n=1}^{\infty}\left(\frac{x}{5}\right)^n = \sum_{n=1}^{\infty}\left(\frac{-5}{5}\right)^n = \sum_{n=1}^{\infty}(-1)^n = -1+1-1+1-1+\ldots$

Hence, when $x = -5, \sum_{n=1}^{\infty}\left(\frac{x}{5}\right)^n$ diverges.

Therefore, Interval of Convergence is $(-5, 5)$.

Find Interval of convergence for $\sum_{n=1}^{\infty} \frac{x^{3n}}{(n+1)!}$.

Let $u_n = \frac{x^{3n}}{(n+1)!}$ and $u_{n+1} = \frac{x^{3(n+1)}}{((n+1)+1)!} = \frac{x^{3n+3}}{(n+2)!}$

Note: $(n+1)! = (n+1)(n)(n-1)(n-2)\cdots(3)(2)(1)$

$(n+2)! = (n+2)(n+1)(n)(n-1)(n-2)(n-3)\cdots(3)(2)(1)$

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{x^{3n+3}}{(n+2)!}\frac{(n+1)!}{x^{3n}}\right| = \lim_{n\to\infty}\left|\frac{x^{3n}\cdot x^3}{x^{3n}}\right| \cdot \lim_{n\to\infty}\left|\frac{(n+1)!}{(n+2)!}\right|$$

$$= \left|x^3\right| \cdot \lim_{n\to\infty}\left|\frac{1}{(n+2)}\right| = \left|x^3\right| \cdot 0 = 0$$

Find Interval of convergence for $\sum_{n=1}^{\infty} \frac{x^{3n}}{(n+1)!}$.   con't

Hence, $\lim_{n\to\infty} \left| \frac{u_{n+1}}{u_n} \right| < 1$ for any x.

$\sum_{n=1}^{\infty} \frac{x^n}{(n+2)!}$ converges for any x.

Therefore Interval of convergence $= (-\infty, \infty)$.

Find Interval of convergence for $\sum_{\mathrm{n=1}}^{\infty}\frac{(x-2)^{n-1}}{2^{n-1}}$.

$$\text{Let } u_n=\frac{(x-2)^{n-1}}{2^{n-1}} \quad \text{and} \quad u_{n+1}=\frac{(x-2)^{(n+1)-1}}{2^{(n+1)-1}}=\frac{(x-2)^n}{2^n}$$

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right|=\lim_{n\to\infty}\left|\frac{(x-2)^n}{2^n}\cdot\frac{2^{n-1}}{(x-2)^{n-1}}\right|$$

$$=\lim_{n\to\infty}\left|\frac{(x-2)^n}{(x-2)^n\cdot(x-2)^{-1}}\right|\cdot\lim_{n\to\infty}\left|\frac{2^n\cdot 2^{-1}}{2^n}\right|$$

$$=\lim_{n\to\infty}\left|\frac{1}{(x-2)^{-1}}\right|\cdot\lim_{n\to\infty}\left|2^{-1}\right|=\lim_{n\to\infty}\left|(x-2)^1\right|\cdot\lim_{n\to\infty}\left|\frac{1}{2}\right|=\frac{1}{2}\left|x-2\right|$$

$$\text{Set } \frac{1}{2}\left|x-2\right|<1 \quad \Rightarrow \quad -1<\frac{1}{2}\left(x-2\right)<1$$

$$\Rightarrow \quad -2<\left(x-2\right)<2 \quad \Rightarrow \quad 0<x<3$$

Find Interval of convergence for $\sum_{n=1}^{\infty} \frac{(x-2)^{n-1}}{2^{n-1}}$. con't

We also to need to test $x=0$ and $x=3$.

For $x=0$, $\sum_{n=1}^{\infty} \frac{(x-2)^{n-1}}{2^{n-1}} = \sum_{n=1}^{\infty} \frac{(0-2)^{n-1}}{2^{n-1}} = \sum_{n=1}^{\infty} \frac{(-2)^{n-1}}{2^{n-1}}$

$$= \sum_{n=1}^{\infty} \frac{(-1\cdot 2)^{n-1}}{2^{n-1}} = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}(2)^{n-1}}{2^{n-1}} = \sum_{n=1}^{\infty} (-1)^{n-1}$$

$$= 1-1+1-1+1-1+\dots$$

Hence, for $x=0$, $\sum_{n=1}^{\infty} \frac{(x-2)^{n-1}}{2^{n-1}}$ diverges.

Find Interval of convergence for $\sum_{n=1}^{\infty} \frac{(x-2)^{n-1}}{2^{n-1}}$. con't

For $x = 3, \sum_{n=1}^{\infty} \frac{(3-2)^{n-1}}{2^{n-1}} = \sum_{n=1}^{\infty} \frac{(1)^{n-1}}{2^{n-1}}$ Alternating Series

$$a_n = \frac{1}{2^{n-1}}; \quad a_{n+1} = \frac{1}{2^n} \quad \Rightarrow \quad a_{n+1} < a_n$$

$$\lim_{n \to \infty} a_n = \lim_{n \to \infty} \frac{1}{2^{n-1}} = 0$$

Hence, For $x = 3, \sum_{n=1}^{\infty} \frac{(3-2)^{n-1}}{2^{n-1}} = \sum_{n=1}^{\infty} \frac{(1)^{n-1}}{2^{n-1}}$ converges.

Therefore, Interval of convergence = (0, 3].

Find the interval of convergence for $\displaystyle\sum_{n=1}^{\infty} \frac{x^{5n+1}}{(5n+1)!}$.

Let $u_n = \dfrac{x^{5n+1}}{(5n+1)!}$ and $u_{n+1} = \dfrac{x^{5(n+1)+1}}{(5(n+1)+1)!} = \dfrac{x^{5n+6}}{(5n+6)!}$

Note: $(5n+1)! = (5n+1)(5n)(5n-1)(5n-2)(5n-3)\cdots(3)(2)(1)$

$(5n+6)! = (5n+6)(5n+5)(5n+4)(5n+2)(5n+1)\cdots(3)(2)(1)$

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{x^{5n+6}}{(5n+6)!}\,\frac{(5n+1)!}{x^{5n+1}}\right| = \lim_{n\to\infty}\left|\frac{x^{5n}x^{6}}{x^{5n}x^{1}}\right|\cdot\lim_{n\to\infty}\left|\frac{(5n+1)!}{(5n+6)!}\right|$$

$$= \lim_{n\to\infty}\left|x^5\right|\cdot\lim_{n\to\infty}\left|\frac{1}{(5n+6)(5n+5)(5n+4)(5n+2)}\right| = \left|x^5\right|\cdot 0 = 0$$

Find the interval of convergence for $\sum_{n=1}^{\infty} \frac{x^{5n+1}}{(5n+1)!}$. con't

Hence, $\lim_{n\to\infty} \left| \frac{u_{n+1}}{u_n} \right| < 1$ for any x.

Therefore, interval of convergence $\left(-\infty, \infty\right)$