Calculus II Section 9.9 Notes

Objective: Write power series representation of function $f(x)$.

Recall: Geometric Series $\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}$ for $-1 < r < 1$.

Examples:

$$\sum_{n=0}^{\infty} 2\left(\frac{1}{3}\right)^n = 2 + 2\left(\frac{1}{3}\right) + 2\left(\frac{1}{3}\right)^2 + 2\left(\frac{1}{3}\right)^3 + \cdots = \frac{a}{1-r} = \frac{2}{1-\frac{1}{3}} = \frac{2}{2/3} = 3$$

$$\sum_{n=0}^{\infty} 4\left(\frac{1}{5}\right)^n = 4 + 4\left(\frac{1}{5}\right) + 4\left(\frac{1}{5}\right)^2 + 4\left(\frac{1}{5}\right)^3 + \cdots = \frac{a}{1-r} = \frac{4}{1-\frac{1}{5}} = \frac{4}{4/5} = 5$$

$$\sum_{n=0}^{\infty} \left(\frac{1}{6}\right)^n = 1 + \left(\frac{1}{6}\right) + \left(\frac{1}{6}\right)^2 + \left(\frac{1}{6}\right)^3 + \cdots = \frac{a}{1-r} = \frac{1}{1-\frac{1}{6}} = \frac{1}{5/6} = 6/5$$

Example 1: Find a geometric power series for the function $f(x) = \frac{1}{7 - x}$ centered at 0.

Recall: Geometric Series $\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}$ for $-1 < r < 1$.

Note: $\frac{1}{7 - x} = \frac{1/7}{(7 - x)/7} = \frac{1/7}{1 - x/7} = \frac{a}{1-r}$

$a = 1/7$ ; $r = x/7$

Power Series Representation for $f(x)$: $f(x) = \frac{1}{7 - x} = \sum_{n=0}^{\infty} ar^n = \sum_{n=0}^{\infty} \left(\frac{1}{7}\right)\left(\frac{x}{7}\right)^n$

Comparing $f(x) = \dfrac{1}{7-x}$ and $\sum_{n=0}^{\infty}\left(\dfrac{1}{7}\right)\left(\dfrac{x}{7}\right)^n$

For $x = 2$,

$$f(x) = \frac{1}{7-x} = \frac{1}{7-2} = \frac{1}{5}$$

Power Series: $\sum_{n=0}^{\infty}\left(\dfrac{1}{7}\right)\left(\dfrac{x}{7}\right)^n \sum_{n=0}^{\infty}\left(\dfrac{1}{7}\right)\left(\dfrac{2}{7}\right)^n = \dfrac{a}{1-r} = \dfrac{1/7}{1-2/7} = \dfrac{1/7}{5/7} = 1/5$

*Note*: The power series $\sum_{n=0}^{\infty}(1/7)(x/7)^n$ is good when $x$ is around $c = 0$.

Example 2: Find a geometric power series for the function $f(x) = \dfrac{5}{2-x}$ centered at 0.

Note: $\dfrac{5}{2-x} = \dfrac{(5)/2}{(2-x)/2} = \dfrac{5/2}{1-x/2} = \dfrac{a}{1-r}$

$a = 5/2$

$r = x/2$

Power Series Representation for $f(x)$:

$$f(x) = \frac{5}{2-x} = \sum_{n=0}^{\infty} ar^n = \sum_{n=0}^{\infty} \left(\frac{5}{2}\right)\left(\frac{x}{2}\right)^n$$

Comparing $f(x) = \dfrac{5}{2 - x}$ and $\sum_{n=0}^{\infty} ar^n = \sum_{n=0}^{\infty} (5/2)(x/2)^n$

For $x = 1$,

$$f(x) = \frac{5}{2 - x} = \frac{5}{2 - 1} = 5$$

Power Series: $\sum_{n=0}^{\infty} (5/2)(x/2)^n = \sum_{n=0}^{\infty} (5/2)(1/2)^n = \dfrac{a}{1-r} = \dfrac{5/2}{1-1/2} = \dfrac{5/2}{1/2} = 5$

*Note*: The power series $\sum_{n=0}^{\infty} (5/2)(x/2)^n$ is good when $x$ is around $c = 0$.

Example 3: Find a geometric power series for the function $f(x) = \dfrac{2}{5-x}$ centered at $c=1$.

First, rewrite $5-x$ as $5-(x-c)-c$ since we want the power series centered at $c=1$.

Note: $5-x=5-(x-c)-c=5-(x-1)-1= 4-(x-1)$

Hence: $\dfrac{2}{5-x}=\dfrac{2}{5-(x-1)-1}=\dfrac{2}{4-(x-1)}$

$$=\frac{2/4}{4/4-(x-1)/4}=\frac{1/2}{1-(x-1)/4}=\frac{a}{1-r}$$

So we have: $a = \dfrac{1}{2}$ ; $r = \dfrac{x-1}{4}$

Power Series Representation for $f(x)$:

$$f(x)=\frac{2}{5-x}=\sum_{n=0}^{\infty} ar^n=\sum_{n=0}^{\infty}\left(\frac{1}{2}\right)\left[\frac{x-1}{4}\right]^n$$

Comparing $f(x) = \dfrac{2}{5 - x}$ and $\sum_{n=0}^{\infty}\left(\frac{1}{2}\right)\left[\frac{x-1}{4}\right]^n$

Note that $\sum_{n=0}^{\infty}\left(\frac{1}{2}\right)\left[\frac{x-1}{4}\right]^n$ converges if $r = \dfrac{x-1}{4}$ is between -1 and 1.

Hence, $-1 < \dfrac{x-1}{4} < 1 \Leftrightarrow -4 < x-1 < 4 \Leftrightarrow -3 < x < 5$

Or $\sum_{n=0}^{\infty}\left(\frac{1}{2}\right)\left[\frac{x-1}{4}\right]^n$ converges if $-3 < x < 5$ (this interval is centered at 1).

For $x = 1.5$,

$$f(x) = \frac{5}{2 - x} = \frac{5}{2 - 1.5} = 10$$

Power Series: $\sum_{n=0}^{\infty}\left(\frac{1}{2}\right)\left[\frac{x-1}{4}\right]^n = \sum_{n=0}^{\infty}\left(\frac{1}{2}\right)\left[\frac{1.5-1}{4}\right]^n = \sum_{n=0}^{\infty}\left(\frac{1}{2}\right)\left[\frac{0.5}{4}\right]^n$

$$= \frac{1/2}{1-0.5/4} = 10$$

Example 4: Find a geometric power series for the function $f(x) = \dfrac{4}{3x - 6}$ centered at $c = -2$.

Recall: Geometric Series $\sum_{n=0}^{\infty} ar^n = \dfrac{a}{1-r}$ for $-1 < r < 1$.

Since we want a power series centered at $c = -2$, rewrite $3x - 6$ as $3(x - c) - 6 + 3c$.

Note: $3x - 6 = 3(x - c) - 6 + 3c = 3(x - -2) - 6 + 3(-2) = 3(x+2) - 12 = -12 + 3(x+2)$

Note: $\dfrac{4}{3x - 6} = \dfrac{4}{3(x+2) - 6 - 6} = \dfrac{4}{-12 + 3(x + 2)} = \dfrac{4/(-12)}{-12/(-12) + 3(x + 2)/(-12)}$

$$= \frac{\frac{-1}{3}}{1 - \frac{x+2}{4}} = \frac{a}{1-r}$$

Hence, $a = \dfrac{-1}{3}$ and $r = \dfrac{x+2}{4}$

Hence, $a = -1/3$; $r = \dfrac{(x + 2)}{4}$

Power Series Representation for $f(x)$: $f(x) = \dfrac{4}{3x - 6} = \sum_{n=0}^{\infty} \left(\dfrac{-1}{3}\right)\left[\dfrac{(x + 2)}{4}\right]^n$

Comparing $f(x) = \dfrac{4}{3x - 6}$ and $\displaystyle\sum_{n=0}^{\infty} \left(\frac{-1}{3}\right)\left[\frac{(x+2)}{4}\right]^n$

Let $x = -1$.

$$f(x) = \frac{4}{3x - 6} = \frac{4}{3(-1) - 6} = -\frac{4}{9}$$

$$\sum_{n=0}^{\infty} \left(\frac{-1}{3}\right)\left[\frac{(x+2)}{4}\right]^n = \sum_{n=0}^{\infty} \left(\frac{-1}{3}\right)\left[\frac{(-1+2)}{4}\right]^n = \sum_{n=0}^{\infty} \left(\frac{-1}{3}\right)\left[\frac{1}{4}\right]^n$$

$$= \frac{a}{1-r} = \frac{-1/3}{1-1/4} = \frac{-1/3}{3/4} = -4/9$$

*Note*: The power series $\left(\dfrac{-1}{3}\right)\left[\dfrac{(x+2)}{4}\right]^n$ is good when $x$ is around $c = -2$.

Example 5: Find a power series for the function $f(x) = \dfrac{4x}{x^2+5x+4}$ centered at $c=0$:

Decomposing (see below): $\dfrac{4x}{x^2+5x+4} = \dfrac{4x}{(x+4)(x+1)} = \dfrac{A}{x+4} + \dfrac{B}{x+1} = \dfrac{16/3}{x+4} + \dfrac{-4/3}{x+1}$

$$\frac{16/3}{x+4} = \frac{16/3}{4+x} = \frac{(16/3)/4}{4/4+x/4} = \frac{4/3}{1-(-x/4)} = \frac{a}{1-r}$$

$$\frac{-4/3}{x+1} = \frac{-4/3}{1+x} = \frac{-4/3}{1-(-x)} = \frac{a}{1-r}$$

Power Series Representation for $f(x)$:

$$f(x) = \frac{4x}{x^2+5x+4} = \sum_{n=0}^{\infty}(4/3)(-x/4)^n + \sum_{n=0}^{\infty}(-4/3)(-x)^n$$

$$= \sum_{n=0}^{\infty}\left[(4/3)(-x/4)^n + (-4/3)(-x)^n\right]$$

Using Partial Fraction To Decompose $\frac{4x}{x^2+5x+4}$:

$$\frac{4x}{x^2+5x+4} = \frac{4x}{(x+4)(x+1)} = \frac{A}{x+4} + \frac{B}{x+1}$$

Hence, $\frac{4x}{(x+4)(x+1)} = \frac{A(x+1)+B(x+4)}{(x+4)(x+1)}$

$$4x = A(x+1) + B(x+4)$$

Let $x = -1 \Rightarrow 4(-1) = A(-1+1) + B(-1+4)$

$$-4 = 3B \Rightarrow B = -4/3$$

Let $x = -4 \Rightarrow 4(-4) = A(-4+1) + B(-4+4)$

$$-16 = -3A \Rightarrow A = 16/3$$