Calculus III

Test 1

Test 1

① $\underset{\sim}{u} \times \underset{\sim}{v} = \begin{bmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ 6 & -5 & 2 \\ -4 & 2 & 3 \end{bmatrix} = \langle -19, -26, -8 \rangle$

② $\vec{PQ} = \langle 8 - {}^{-}1, 10-4, 5-3 \rangle = \langle 9, 6, 2 \rangle$

$\vec{PR} = \langle -2 - {}^{-}1, 3-4, 1-3 \rangle = \langle -1, -1, {}^{-}2 \rangle$

$\underset{\sim}{n} = \vec{PQ} \times \vec{PR} = \langle -10, 16, -3 \rangle$

Equ. of Plane (using $\underset{\sim}{n}$ and P):

$$a(x-x_1) + b(y-y_1) + c(z-z_1) = 0$$

$$-10(x - {}^{-}1) + 16(y-4) + {}^{-}3(z-3) = 0$$

$$-10(x+1) + 16(y-4) - 3(z-3) = 0$$

③ $x = 6\cos(-\pi/4) = 3\sqrt{2}$

$y = 6\sin(-\pi/4) = -3\sqrt{2}$

$z = 2$

④ $r = \sqrt{x^2+y^2} = \sqrt{(3)^2 + (-3)^2} = 3\sqrt{2}$

Test 1

⑤ $r^2 = x^2 + y^2$

$z = x^2 + y^2 - 11 = r^2 - 11$

Ans: $z = r^2 - 11$

⑥ $\rho^2 = x^2 + y^2 + z^2$

$z^2 = \rho^2 \cos^2\phi$

So $x^2 + y^2 - 3z^2 = 0$

$x^2 + y^2 + z^2 - z^2 - 3z^2 = 0$

$\rho^2 - 4z^2 = 0$

$\rho^2 - 4\rho^2 \cos^2\phi = 0$

$\rho^2(1 - 4\cos^2\phi) = 0$

$\cos^2\phi = \frac{1}{4}$

$\cos\phi = \frac{1}{2}$

⑦ $\overrightarrow{PQ} = \langle 8 - -1, 10 - 4, 5 - 3 \rangle = \langle 9, 6, 2 \rangle$

Parametric Equations: $x = x_1 + at$
$y = y_1 + bt$
$z = z_1 + ct$

$\Rightarrow$ $x = -1 + 9t$
$y = 4 + 6t$
$z = 3 + 2t$

Test 1

(8) On calculator, use Parametric Equ. Graphing

$$x = t + 2$$
$$y = t^2 - 1$$

(9) $\underset{\sim}{r}'(t) = \langle 4, 2t, 4t \rangle$

$\underset{\sim}{r}''(t) = \langle 0, 2, 4 \rangle$

$\underset{\sim}{r}'(t) \cdot \underset{\sim}{r}''(t) = 4(0) + (2t)(2) + (4t)(4)$

$= 4t + 16t = 20t$

$\underset{\sim}{r}'(t) \times r''(t) = \big((2t)(4) - (4t)(2)\big)\underset{\sim}{i}$

$- \big((4)(4) - (0)(4t)\big)\underset{\sim}{j}$

$+ \big((4)(2) - (0)(2)\big)\underset{\sim}{k}$

$= \langle 0, -16, 8 \rangle$

(10) $\langle -\cos t, \sin t, \frac{1}{2}e^{2t} \rangle + \underset{\sim}{C}$

(11) $\underset{\sim}{r}(t) = \langle t, t^2, \frac{2}{3}t^3 \rangle$; $P(2, 4, 16/3)$

$\underset{\sim}{r}'(t) = \langle 1, 2t, 2t^2 \rangle$

At $P(2, 4, 16/3)$; $t = 2$

$\underset{\sim}{r}'(2) = \langle 1, 4, 8 \rangle$; $\|\underset{\sim}{r}(2)\| = \sqrt{1^2 + 4^2 + 8^2} = 9$

$$\underset{\sim}{T}(2) = \frac{\underset{\sim}{r}'(2)}{\|\underset{\sim}{r}'(2)\|} = \frac{\langle 1, 4, 8 \rangle}{9}$$

$$= \langle 1/9, 4/9, 8/9 \rangle$$

Parametric Equations : $x = t + 2$
$y = 4t + 4$
$z = 8t + 16/3$

(12) $\underset{\sim}{r}(t) = \langle t^2, 2t \rangle$ $\qquad \underset{\sim}{r}'(t) = \langle 2t, 2 \rangle$

$\|\underset{\sim}{r}'(t)\| = \sqrt{4t^2 + 4}$

$$s = \int_a^b \|\underset{\sim}{r}'(t)\|\, dt = \int_0^3 \sqrt{4t^2 + 4}\, dt = 11.3053$$

Test ~~4~~

(13) $\underset{\sim}{r}(t) = \langle 2\sqrt{t}, 3t, 0\rangle$

$\underset{\sim}{r}'(t) = \langle 1/\sqrt{t}, 3, 0\rangle$

$\|\underset{\sim}{r}'(t)\| = \sqrt{\left(\frac{1}{\sqrt{t}}\right)^2 + 3^2} = \sqrt{\frac{1}{t} + 9}$

$r''(t) = \langle -\frac{1}{2} t^{-3/2}, 0, 0\rangle$

$\underset{\sim}{r}' \times \underset{\sim}{r}'' = \langle 0, 0, \frac{3}{2} t^{-3/2}\rangle$

$\|\underline{r}' \times r''\| = \frac{3}{2t^{3/2}}$

$K = \frac{\|\underset{\sim}{r}'(t) \times \underline{r}''(t)\|}{\|\underset{\sim}{r}'(t)\|^3} = \frac{3}{2(1+9t)^{3/2}}$

(14) $\underset{\sim}{r}(t) = \langle 4\cos t, 3\sin t, t\rangle$ $\quad P(-4, 0, \pi\rangle \Rightarrow t = \pi$

$\underset{\sim}{r}'(t) = \langle -4\sin t, 3\cos t, 1\rangle$

$\underline{r}''(t) = \langle -4\cos t, -3\sin t, 0\rangle$

$\underset{\sim}{r}''(\pi) = \langle 4, 0, 0\rangle$

$\underset{\sim}{r}' \times \underset{\sim}{r}'' = \langle 0, 4, 12\rangle$

$K = \frac{\|\underline{r}' \times \underline{r}''\|}{\|r'\|^3} = 2/5$

Calculus III

Test 2

Test 2

① $\frac{\partial f}{\partial x} = 8x - 2y$

$\frac{\partial f}{\partial y} = -2x + 2y$

② $\frac{\partial f}{\partial x} = \frac{1}{x^2+y^2+1}(2x)$

$\frac{\partial f}{\partial y} = \frac{1}{x^2+y^2+1}(2y)$

③ $f_x = -\sin(x-2y)$

$f_{xx} = -\cos(x-2y)$

$f_y = -\sin(x-2y)(-2)$

$f_{yy} = 2\cos(x-2y)$

$f_{xy} = -\cos(x-2y)(-2)$

Test 2

(4) $\dfrac{dw}{dt} = \dfrac{\partial w}{\partial x} \cdot \dfrac{\partial x}{\partial t} + \dfrac{\partial w}{\partial y} \cdot \dfrac{\partial y}{\partial t}$

$= (-1)(-\sin t) + (2y)(\cos t)$

$= \sin t + 2 \sin t \cos t$

(5) $\dfrac{\partial w}{\partial r} = \dfrac{\partial w}{\partial x} \cdot \dfrac{\partial x}{\partial r} + \dfrac{\partial w}{\partial y} \cdot \dfrac{\partial y}{r} + \dfrac{\partial w}{\partial z} \cdot \dfrac{\partial z}{\partial r}$

$= (2x)(\cos t) + (2y)\sin t + (2z)(0)$

$\dfrac{\partial w}{\partial t} = \dfrac{\partial w}{\partial x} \cdot \dfrac{\partial x}{\partial t} + \dfrac{\partial w}{\partial y} \cdot \dfrac{\partial y}{\partial t} + \dfrac{\partial w}{\partial z} \cdot \dfrac{\partial z}{\partial t}$

$= (2x)(-r \sin t) + (2y)(r\cos t) + 2z$

(6) $\underset{\sim}{u} = \dfrac{\underset{\sim}{v}}{\|\underset{\sim}{v}\|} = \left\langle \dfrac{2}{\sqrt{5}}, \dfrac{1}{\sqrt{5}} \right\rangle$; Note: $\|\underset{\sim}{v}\| = \sqrt{1^2+2^2} = \sqrt{5}$

$\dfrac{\partial f}{\partial x} = -2x$; $\dfrac{\partial f}{\partial y} = \frac{1}{2} y$

$D_{\underset{\sim}{u}} f(x,y) = \dfrac{\partial f}{\partial x} \cos\theta + \dfrac{\partial f}{\partial y} \sin\theta$

$= -2x \cos\theta + \frac{1}{2} y \sin\theta$

$D_{\underset{\sim}{u}} f(1,4) = \dfrac{-2}{\sqrt{5}}$

Test 2

#7 $F(x,y,z) = 9x^2 + y^2 + 4z^2 - 25 = 0$

$P(0,-3,2)$

$\nabla F = \langle 18x, 2y, 8z \rangle$

$\nabla F(0,-3,2) = \langle 0, -6, 16 \rangle$

Tangent Plane: $0(x-0) + -6(y - -3) + 16(z-2) = 0$

$-6y + 16z = 50$

8 $F(x,y,z) = x^2 + y^2 + z^2 - 9 = 0$

$\nabla F = \langle 2x, 2y, 2z \rangle$

$\nabla F(1,2,2) = \langle 2, 4, 4 \rangle$

Equ. of Tangent Plane:

$2(x-1) + 4(y-2) + 4(z-2) = 0$

Equ. of Normal Line:

$$\frac{x-1}{1} = \frac{y-2}{2} = \frac{z-2}{2}$$

Test 2

(9) $f_x = 2x + 3y - 5 = 0$ Eq. 1

$f_y = 3x + 2y = 0$ Eq. 2

Solve system of equations $\Rightarrow$ $x = -2$
$y = 3$

$f_{xx} = 2$

$f_{yy} = 2$

$f_{xy} = 3$

$d = [f_{xx}][f_{yy}] - [f_{xy}]^2 = 4 - 9 = -5$

Thus, $(-2, 3)$ is a saddle point

(10) $\nabla f = \lambda \nabla g$

$\langle 6x, -2y \rangle = \lambda \langle 2, -2 \rangle$

$\Rightarrow 6x = 2\lambda$ (1)

$-2y = -2\lambda$ (2)

(1) $\lambda = 3x$

(2) $-2y = -2(3x) = -6x$

$6x = 2y$

$y = 3x$

$g(x, y) = 2x - 2y + 5 = 0$

$2x - 2(3x) + 5 = 0$

$x = 5/4$ $\Rightarrow y = 15/4$

Test 2

(11) $$I = \int_0^2 \underbrace{\int_{x^2}^{2x} (x^2+2y)\,dy}_{I_1}\,dx$$

$$I_1 = \left(x^2y \cdot 2\cdot\frac{y^2}{2}\right) = x^2y + y^2 \Big|_{x^2}^{2x}$$

$$= \left[x^2(2x) + (2x)^2\right]$$

$$- \left[x^2\cdot x^2 + (x^2)^2\right]$$

$$= 3x^3 + 4x^2 - 2x^4$$

$$I = \int_0^2 \left(3x^3 + 4x^2 - 2x^4\right)dx$$

$$= \frac{3x^4}{4} + \frac{4x^3}{3} - \frac{2x^5}{5}\Big|_0^2$$

$$= \frac{3(16)}{4} + \frac{4(8)}{3} - \frac{2(32)}{5}$$

Test 2

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$$I = \int_0^1 \underbrace{\int_0^{2Y} (\cancel{9+} 3x^2 + 3y^2)\, dx}_{I_1} dy$$

$$I_1 = \int (9 + 3x^2 + 3Y^2)\, dx$$

$$= 9x + \frac{3x^3}{3} + 3Y^2 x$$

$$= 9x + x^3 + 3Y^2 x \Big|_0^{2Y}$$

$$= 9(2Y) + (2Y)^3 + 3Y^2 \cdot (2Y)$$

$$= 18Y + 8Y^3 + 6Y^3$$

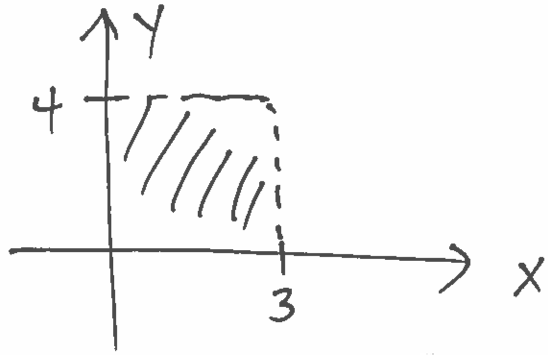
$$I = \int_0^1 (18Y + 8Y^3 + 6Y^3)\, dY$$

$$= \frac{18Y^2}{2} + \frac{8Y^4}{4} + \frac{6Y^4}{4} \Big|_0^1$$

$$= \frac{18}{2} + \frac{8}{4} + \frac{6}{4}$$

Test 2

(13) Region R



$$\text{Volume} = \int_0^4 \int_0^3 (6-y)\,dx\,dy$$

$$= \int_0^4 (6x - yx) \Big|_0^3 \, dy$$

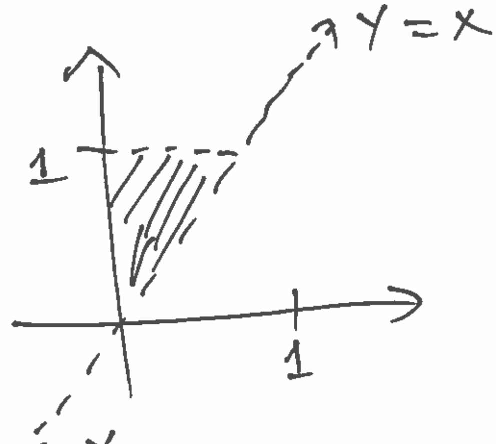
$$= \int_0^4 (18 - 3y)\,dy = 18y - \frac{3y^2}{2} \Big|_0^4$$

$$= 32 - \frac{3(16)}{2}$$

(14)

Test 2

Region R:



$$\text{Volume} = \int_0^1 \underbrace{\int_{x=0}^{x=y} (1 - xy)\, dx}_{I_1}\, dy$$

$$I_1 = \int_0^y (1 - xy)\, dx$$

$$= \left[x - \frac{x^2}{2} \cdot y \right]_0^y = y - \frac{y^3}{2}$$

$$I = \int_0^1 \left(y - y^3/2 \right) dy$$

$$= \frac{y^2}{2} - \frac{y^4}{2 \cdot 4} = \frac{y^2}{2} - \frac{y^4}{8} \Big|_0^1$$

$$= \frac{1}{2} - \frac{1}{8}$$

Test 3

Name Key

Date

Calculus III Test 3

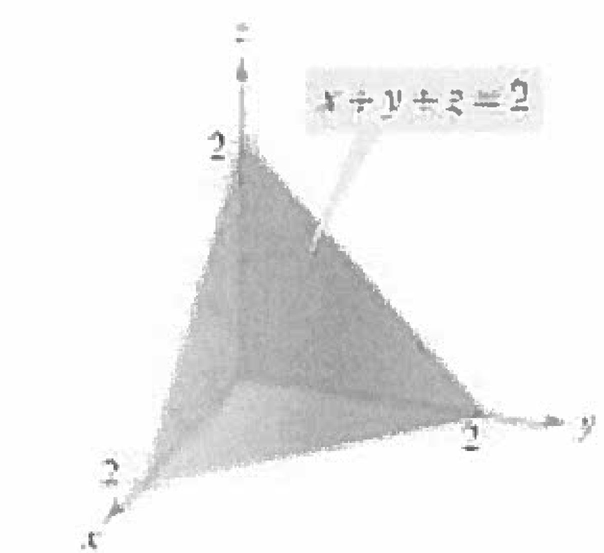
1) Find $\int_0^2 \int_{x^2}^{2x} \left(x^2+2y\right) dy\,dx$

Answer = $88/15$

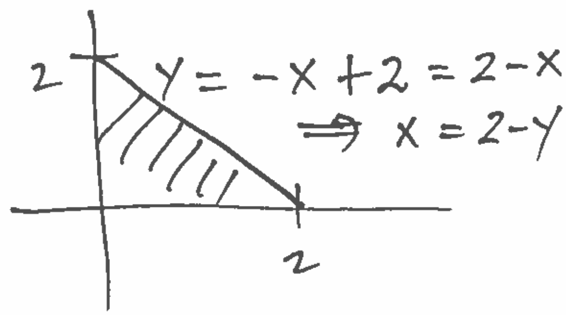
2) Use a double integral to find the volume of the solid.

a) Set up double integral: $\int_0^2 \int_0^{2-x} (2-x-y)\,dy\,dx$

b) Volume = $4/3$



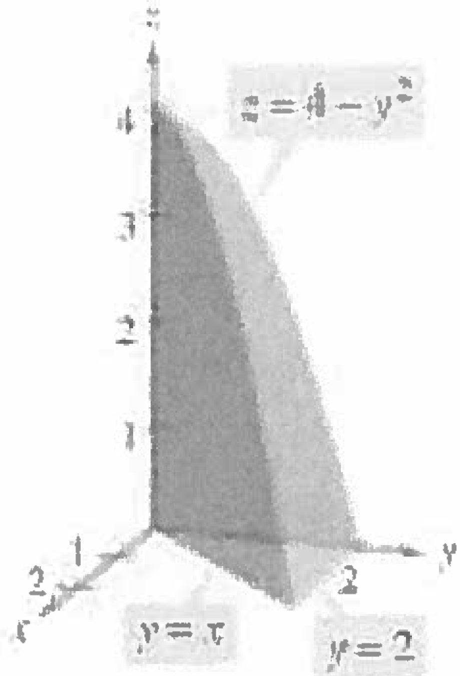
Region R:



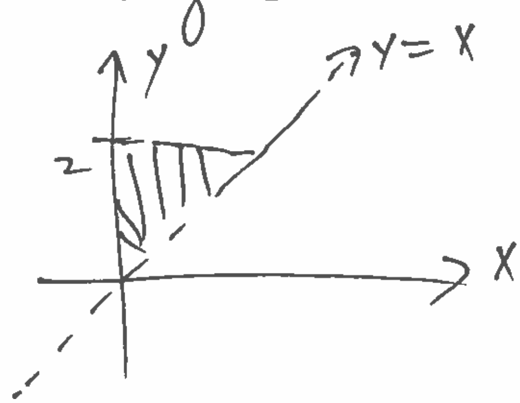
3) Use a double integral to find the volume of the solid.

a) Set up double integral: $\int_0^2 \int_x^2 (4-y^2)\,dy\,dx$

b) Volume = 4



Region R:



4) Find the mass and center of mass of the lamina bounded by the graphs of the equations for the given density.

$y = 6 - x;\ \ y = 0;\ \ x = 0;\ \ \rho = kx^2$

$m =$? $108k$ Hint: $m = \iint \rho(x,y)\,dA$

$M_x =$? $\frac{648}{5}k$ Hint: $M_x = \iint \rho(x,y)\,y\,dA$

$M_y =$? $\frac{1944}{5}k$ Hint: $M_y = \iint \rho(x,y)\,x\,dA$

$\bar{x} =$? $18/5$

$\bar{y} =$? $6/5$

5) Find the surface area of $z = f(x,y)$ over region R.

$f(x,y) = 2x + y^2;\quad R$: triangle with vertices (0,0), (2,0), (2,2)

Hint: Surface Area $= \iint_R \sqrt{1 + [f_x(x,y)]^2 + [f_y(x,y)]^2}\,dA$

a) Set up double integral: $\int_0^2 \int_0^x \sqrt{1 + (2)^2 + (2y)^2}\,dy\,dx$ ✓

b) Surface Area = optional

6) Evaluate the double integral $\int_R \int 25xy\,dA$ by using Jacobian Transformation.

Hint: $\int_R \int f(x,y)\,dx\,dy = \int_S \int f(g(u,v),h(u,v))\,|J| \cdot du \cdot dv$

Use the indicated change of variables to evaluate the double integral.

$x = \frac{1}{2}(u+v)$; $\qquad y = -\frac{1}{2}(u-v)$

a) $u =$? $\quad x - y$

b) $v =$? $\quad x + y$

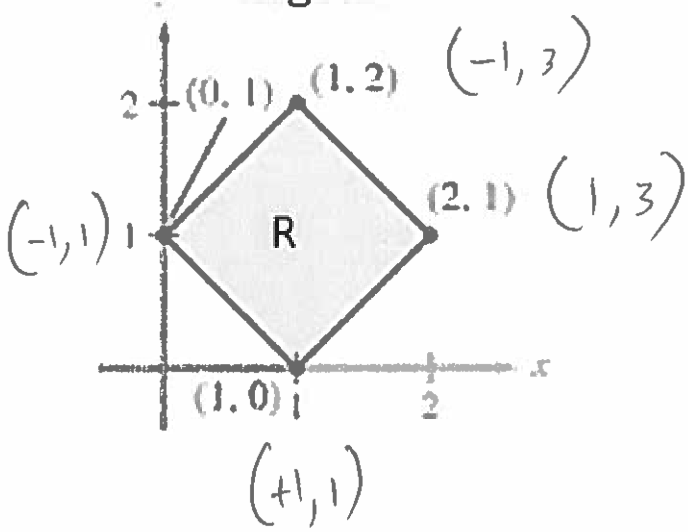
c) Draw Region S

d) $J = \dfrac{\partial(x,y)}{\partial(u,v)} = \dfrac{\partial(x)}{\partial(u)} \cdot \dfrac{\partial(y)}{\partial(v)} - \dfrac{\partial(y)}{\partial(u)} \cdot \dfrac{\partial(x)}{\partial(v)} =$? $\quad 1/2$

e) $\int_R \int 25xy\,dA = \int_S \int f(g(u,v),h(u,v))\,|J| \cdot du \cdot dv =$? $\quad \int_{-1}^{1}\int_{1}^{3} 25xy\,dv\,du$

$-\frac{25}{8}\int\int (v^2-u^2)\,dv\,du$

Region R



Draw Region S

$= -\frac{25}{8}\left[\frac{v^3}{3} - u^2 v\right]_1^3$

$= -\frac{25}{8}\left[9 - 3u^2 - \frac{1}{3} + u^2\right]$

$= -\frac{25}{8}\left[\frac{26}{3} - 2u^2\right]$

$= -\frac{25}{8}\left[\frac{26}{3}u - \frac{2}{3}u^3\right]_{-1}^{1}$

$= -\frac{25}{8}\left[\frac{26}{3} - \frac{2}{3} + \frac{26}{3} - \frac{2}{3}\right]$

$= -\frac{25}{8}\left[\frac{52}{3} - \frac{4}{3}\right] = -50$

7) Find $\int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1} x^2 y^2 z^2 \, dx\,dy\,dz$

Answer = $8/27$

8) Find a potential function for the vector field.

$\mathbf{F}(x, y) = \langle 3x^2y^2,\ 2x^3y \rangle$

$f_x =$ $3x^2y^2$

$f_y =$ $2x^3y$

$f = \int f_x \, dx = x^3y^2$

$f = \int f_y \, dy = x^3y^2$

Potential Function $f(x, y) =$ x^3y^2

9) Find a potential function for the vector field.

$\mathbf{F}(x, y, z) = \langle y^2z^3,\ 2xyz^3,\ 3xy^2z^2 \rangle$

$f_x =$ y^2z^3

$f_y =$ $2xyz^3$

$f_z =$ $3xy^2z^2$

$\underset{\sim}{F} = \nabla f = \langle f_x, f_y, f_z \rangle$

$f = \int f_x = xy^2z$

$f = \int f_y = 2x\frac{y^2}{2}z^3 = xy^2z^3$

$f = \int f_z = 3xy^2\frac{z^3}{3}$

$= xy^2z^3$

Potential Function $f(x, y, z) =$ xy^2z^3

10) Find $\int_C xy\,ds$ along path C.

$C: x = 5t;\ y = 4t;\ 0 \le t \le 1$

Answer = $\frac{20\sqrt{41}}{3}$

11) Evaluate $\int_C \mathbf{F}\cdot d\mathbf{r}$.

$\mathbf{F}(x,y) = \langle x-y, x+y\rangle = \langle x(t), y(t)\rangle$

$C: \mathbf{r}(t) = \langle 4\cos t, 3\sin t\rangle \qquad 0 \le t \le 1$

$x(t), y(t)$

Answer = 9.5

$$\frac{d\mathbf{r}}{dt} = \mathbf{r}'(t) = \langle -4\sin t, \ 3\cos t\rangle$$

$$\mathbf{F}\cdot d\mathbf{r} = \langle 4\cos t - 3\sin t, \ 4\cos t + 3\sin t\rangle \cdot \langle -4\sin t, \ 3\cos t\rangle$$

$$= -16\cos t\sin t + 12\sin^2 t + 12\cos^2 + 9\cos t\sin t$$

$$= -7\cos t\sin t + 12$$

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \int_0^1 (-7\cos t\sin t + 12)\,dt = 9.5$$