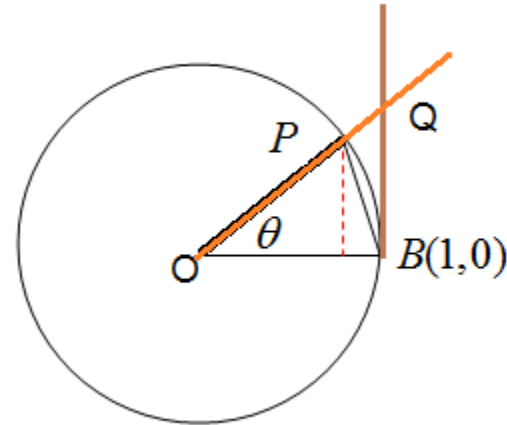
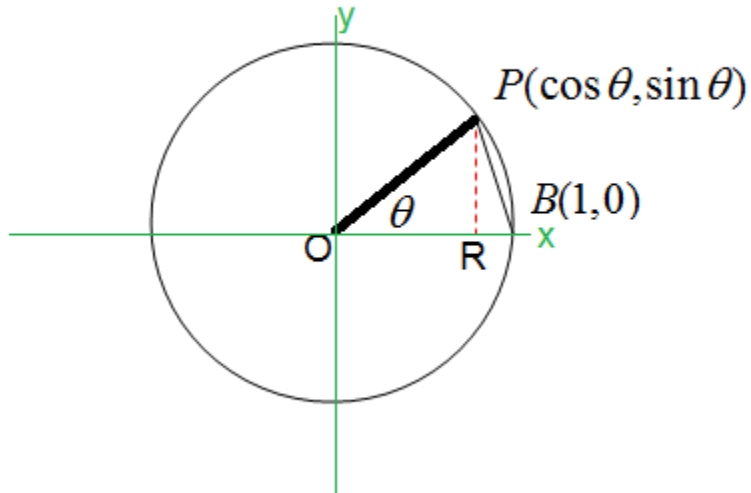


Claim: $\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1$

Proof: Unit Circle with $r = 1$



Note: $\sin \theta = \frac{\text{Length of PR}}{\text{Length of OP}} = \frac{\text{Length of PR}}{1} = \text{Length of PR}$

Note: $\tan \theta = \frac{\text{Length of BQ}}{\text{Length of OB}} = \frac{\text{Length of BQ}}{1} = \text{Length of BQ}$

Area of $\triangle OBP = \frac{1}{2}(\text{base})(\text{height}) = \frac{1}{2}(\text{length of OB})(\text{Length of PR}) = \frac{1}{2}(1)(\sin \theta) = \frac{1}{2} \sin \theta$

Area of Sector OBP = $\frac{1}{2}(\text{radius of circle})^2 (\theta) = \frac{1}{2}(1)^2 (\theta) = \frac{1}{2} \theta$

Area of $\triangle OBQ = \frac{1}{2}(\text{base})(\text{height}) = \frac{1}{2}(\text{length of OB})(\text{Length of BQ}) = \frac{1}{2}(1)(\tan \theta) = \frac{1}{2} \tan \theta$

Note: $0 < \text{Area of } \triangle OBP < \text{Area of Sector OBP} < \text{Area of } \triangle OBQ$

$$0 < \frac{1}{2} \sin \theta < \frac{1}{2} \theta < \frac{1}{2} \tan \theta$$

$$0 < \sin \theta < \theta < \tan \theta \quad \text{Multiply each term by 2}$$

$$0 < 1 < \frac{\theta}{\sin \theta} < \frac{\tan \theta}{\sin \theta} \quad \text{Divide each term by } \sin \theta;$$

$$0 < 1 < \frac{\theta}{\sin \theta} < \frac{1}{\cos \theta} \quad \text{Note: } \frac{\tan \theta}{\sin \theta} = \frac{\sin \theta / \cos \theta}{\sin \theta} = \frac{1}{\cos \theta}$$

$$1 > \frac{\sin \theta}{\theta} > \frac{\cos \theta}{1} \quad \text{Taking reciprocals}$$

$$1 > \frac{\sin \theta}{\theta} > \cos \theta$$

$$\text{Hence, } \lim_{\theta \rightarrow 0} 1 > \lim_{\theta \rightarrow 0} \frac{\sin \theta}{\theta} > \lim_{\theta \rightarrow 0} \cos \theta$$

$$1 > \lim_{\theta \rightarrow 0} \frac{\sin \theta}{\theta} > 1 \quad \text{Recall } \lim_{\theta \rightarrow 0} \cos \theta = 1.$$

Therefore, $\lim_{\theta \rightarrow 0} \frac{\sin \theta}{\theta} = 1$ by The Squeeze (or Sandwich) Theorem.

Claim: $\lim_{x \rightarrow 0} \frac{1 - \cos x}{x} = 0$

Proof:

Note: $\sin^2 x + \cos^2 x = 1 \Rightarrow \sin^2 x = 1 - \cos^2 x$

$$\frac{1 - \cos x}{x} = \frac{1 - \cos x}{x} \cdot \frac{1 + \cos x}{1 + \cos x} = \frac{1 - \cos^2 x}{x(1 + \cos x)} = \frac{\sin^2 x}{x(1 + \cos x)} = \frac{\sin x \cdot \sin x}{x(1 + \cos x)} = \frac{\sin x}{x} \cdot \frac{\sin x}{(1 + \cos x)}$$

Hence, $\lim_{x \rightarrow 0} \frac{1 - \cos x}{x} = \lim_{x \rightarrow 0} \left[\frac{\sin x}{x} \cdot \frac{\sin x}{(1 + \cos x)} \right] = \lim_{x \rightarrow 0} \left[\frac{\sin x}{x} \right] \cdot \lim_{x \rightarrow 0} \left[\frac{\sin x}{(1 + \cos x)} \right] = (1) \left(\frac{0}{1} \right) = 0$

Claim: $\lim_{x \rightarrow 0} \frac{\cos x - 1}{x} = 0$

Proof:

$$\frac{\cos x - 1}{x} = \frac{-1(\cos x - 1)}{-1x} = \frac{1 - \cos x}{-1x} = \frac{1}{-1} \cdot \frac{1 - \cos x}{x} = (-1) \cdot \frac{1 - \cos x}{x}$$

Hence, $\lim_{x \rightarrow 0} \frac{\cos x - 1}{x} = \lim_{x \rightarrow 0} \left[(-1) \cdot \frac{1 - \cos x}{x} \right] = \lim_{x \rightarrow 0} [(-1)] \cdot \lim_{x \rightarrow 0} \left[\frac{1 - \cos x}{x} \right] = (-1)(0) = 0$

Claim: $D_x(\sin x) = \cos x$

Recall: $\sin x(A + B) = \sin A \cdot \cos B + \cos A \cdot \sin B$

$$\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1; \quad \lim_{x \rightarrow 0} \frac{1 - \cos x}{x} = 0; \quad \lim_{x \rightarrow 0} \frac{\cos x - 1}{x} = 0$$

Proof:

$$\begin{aligned} D_x(\sin x) &= \lim_{\Delta x \rightarrow 0} \left[\frac{\sin(x + \Delta x) - \sin x}{\Delta x} \right] = \lim_{\Delta x \rightarrow 0} \left[\frac{\sin x \cdot \cos \Delta x + \cos x \cdot \sin \Delta x - \sin x}{\Delta x} \right] \\ &= \lim_{\Delta x \rightarrow 0} \left[\frac{\cos x \cdot \sin \Delta x + \sin x \cdot \cos \Delta x - \sin x}{\Delta x} \right] = \lim_{\Delta x \rightarrow 0} \left[\frac{\cos x \cdot \sin \Delta x + \sin x \cdot (\cos \Delta x - 1)}{\Delta x} \right] \\ &= \lim_{\Delta x \rightarrow 0} \left[\frac{\cos x \cdot \sin \Delta x}{\Delta x} + \frac{\sin x \cdot (\cos \Delta x - 1)}{\Delta x} \right] = \lim_{\Delta x \rightarrow 0} \left[\cos x \cdot \frac{\sin \Delta x}{\Delta x} + \sin x \cdot \frac{(\cos \Delta x - 1)}{\Delta x} \right] \\ &= \lim_{\Delta x \rightarrow 0} [\cos x] \cdot \lim_{\Delta x \rightarrow 0} \left[\frac{\sin \Delta x}{\Delta x} \right] - \lim_{\Delta x \rightarrow 0} [\sin x] \cdot \lim_{\Delta x \rightarrow 0} \left[\frac{(\cos \Delta x - 1)}{\Delta x} \right] \\ &= \cos x \cdot [1] - \sin x \cdot [0] = \cos x \end{aligned}$$

Claim: $D_x(\cos x) = -\sin x$

Recall: $\cos(A + B) = \cos A \cdot \cos B - \sin A \cdot \sin B$

$$\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1; \quad \lim_{x \rightarrow 0} \frac{1 - \cos x}{x} = 0; \quad \lim_{x \rightarrow 0} \frac{\cos x - 1}{x} = 0$$

Proof:

$$\begin{aligned} D_x(\cos x) &= \lim_{\Delta x \rightarrow 0} \left[\frac{\cos(x + \Delta x) - \cos x}{\Delta x} \right] = \lim_{\Delta x \rightarrow 0} \left[\frac{\cos x \cdot \cos \Delta x - \sin x \cdot \sin \Delta x - \cos x}{\Delta x} \right] \\ &= \lim_{\Delta x \rightarrow 0} \left[\frac{\cos x \cdot \cos \Delta x - \cos x - \sin x \cdot \sin \Delta x}{\Delta x} \right] = \lim_{\Delta x \rightarrow 0} \left[\frac{\cos x \cdot (\cos \Delta x - 1) - \sin x \cdot \sin \Delta x}{\Delta x} \right] \\ &= \lim_{\Delta x \rightarrow 0} \left[\frac{\cos x \cdot (\cos \Delta x - 1)}{\Delta x} - \frac{\sin x \cdot \sin \Delta x}{\Delta x} \right] = \lim_{\Delta x \rightarrow 0} \left[\cos x \cdot \frac{(\cos \Delta x - 1)}{\Delta x} - \sin x \cdot \frac{\sin \Delta x}{\Delta x} \right] \\ &= \lim_{\Delta x \rightarrow 0} [\cos x] \cdot \lim_{\Delta x \rightarrow 0} \left[\frac{(\cos \Delta x - 1)}{\Delta x} \right] - \lim_{\Delta x \rightarrow 0} [\sin x] \cdot \lim_{\Delta x \rightarrow 0} \left[\frac{\sin \Delta x}{\Delta x} \right] \\ &= \cos x \cdot [0] - \sin x \cdot [1] = -\sin x \end{aligned}$$

Product Rule for Derivative.

$$\text{Claim: } D_x(f(x) \cdot g(x)) = f(x) \cdot g'(x) + g(x) \cdot f'(x)$$

$$\text{Recall: } g'(x) = \lim_{\Delta x \rightarrow 0} \frac{[g(x + \Delta x) - g(x)]}{\Delta x}; \quad f'(x) = \lim_{\Delta x \rightarrow 0} \frac{[f(x + \Delta x) - f(x)]}{\Delta x}$$

Proof:

$$\begin{aligned} D_x(f(x) \cdot g(x)) &= \lim_{\Delta x \rightarrow 0} \left[\frac{f(x + \Delta x) \cdot g(x + \Delta x) - f(x) \cdot g(x)}{\Delta x} \right] \\ &= \lim_{\Delta x \rightarrow 0} \left[\frac{f(x + \Delta x) \cdot g(x + \Delta x) - f(x + \Delta x) \cdot g(x) + f(x + \Delta x) \cdot g(x) - f(x) \cdot g(x)}{\Delta x} \right] \\ &= \lim_{\Delta x \rightarrow 0} \left[\frac{f(x + \Delta x) \cdot g(x + \Delta x) - f(x + \Delta x) \cdot g(x)}{\Delta x} + \frac{f(x + \Delta x) \cdot g(x) - f(x) \cdot g(x)}{\Delta x} \right] \\ &= \lim_{\Delta x \rightarrow 0} \left[\frac{f(x + \Delta x) \cdot [g(x + \Delta x) - g(x)]}{\Delta x} + \frac{g(x) \cdot [f(x + \Delta x) - f(x)]}{\Delta x} \right] \\ &= \lim_{\Delta x \rightarrow 0} \left[f(x + \Delta x) \cdot \frac{[g(x + \Delta x) - g(x)]}{\Delta x} + g(x) \cdot \frac{[f(x + \Delta x) - f(x)]}{\Delta x} \right] \\ &= \lim_{\Delta x \rightarrow 0} f(x + \Delta x) \cdot \lim_{\Delta x \rightarrow 0} \frac{[g(x + \Delta x) - g(x)]}{\Delta x} + \lim_{\Delta x \rightarrow 0} g(x) \cdot \lim_{\Delta x \rightarrow 0} \frac{[f(x + \Delta x) - f(x)]}{\Delta x} \\ &= f(x) \cdot g'(x) + g(x) \cdot f'(x) \end{aligned}$$

Quotient Rule for Derivative.

$$\text{Claim: } D_x \left(\frac{f(x)}{g(x)} \right) = \frac{g(x) \cdot f'(x) - f(x) \cdot g'(x)}{[g(x)]^2}, \quad g(x) \neq 0$$

$$\text{Recall: } g'(x) = \lim_{\Delta x \rightarrow 0} \frac{[g(x + \Delta x) - g(x)]}{\Delta x}; \quad f'(x) = \lim_{\Delta x \rightarrow 0} \frac{[f(x + \Delta x) - f(x)]}{\Delta x}; \quad \frac{A}{B} - \frac{C}{D} = \frac{AD - BC}{DB}$$

Proof:

$$\begin{aligned} D_x \left(\frac{f(x)}{g(x)} \right) &= \lim_{\Delta x \rightarrow 0} \left[\frac{\frac{f(x + \Delta x)}{g(x + \Delta x)} - \frac{f(x)}{g(x)}}{\Delta x} \right] = \lim_{\Delta x \rightarrow 0} \left[\frac{\frac{f(x + \Delta x) \cdot g(x) - g(x + \Delta x) \cdot f(x)}{g(x + \Delta x) \cdot g(x)}}{\Delta x} \right] \\ &= \lim_{\Delta x \rightarrow 0} \left[\frac{f(x + \Delta x) \cdot g(x) - g(x + \Delta x) \cdot f(x)}{g(x + \Delta x) \cdot g(x)} \cdot \frac{1}{\Delta x} \right] = \lim_{\Delta x \rightarrow 0} \left[\frac{f(x + \Delta x) \cdot g(x) - g(x + \Delta x) \cdot f(x)}{g(x + \Delta x) \cdot g(x) \cdot \Delta x} \right] \\ &= \lim_{\Delta x \rightarrow 0} \left[\frac{f(x + \Delta x) \cdot g(x) - \color{red}{f(x)g(x)} + \color{red}{f(x)g(x)} - g(x + \Delta x) \cdot f(x)}{g(x + \Delta x) \cdot g(x) \cdot \Delta x} \right] \\ &= \lim_{\Delta x \rightarrow 0} \left[\frac{g(x)[f(x + \Delta x) - f(x)] + f(x)[g(x) - g(x + \Delta x)]}{g(x + \Delta x) \cdot g(x) \cdot \Delta x} \right] \end{aligned}$$

$$\begin{aligned}
&= \lim_{\Delta x \rightarrow 0} \left[\frac{g(x)[f(x+\Delta x) - f(x)] - f(x)[g(x+\Delta x) - g(x)]}{g(x+\Delta x) \cdot g(x) \cdot \Delta x} \right] \\
&= \lim_{\Delta x \rightarrow 0} \left[\frac{g(x)[f(x+\Delta x) - f(x)] - f(x)[g(x+\Delta x) - g(x)]}{\Delta x} \right] \cdot \frac{1}{g(x+\Delta x) \cdot g(x)} \\
&= \lim_{\Delta x \rightarrow 0} \left[\frac{g(x)[f(x+\Delta x) - f(x)]}{\Delta x} - \frac{f(x)[g(x+\Delta x) - g(x)]}{\Delta x} \right] \cdot \frac{1}{g(x+\Delta x) \cdot g(x)} \\
&= \lim_{\Delta x \rightarrow 0} \left[g(x) \cdot \frac{[f(x+\Delta x) - f(x)]}{\Delta x} - \frac{f(x) \cdot [g(x+\Delta x) - g(x)]}{\Delta x} \right] \cdot \frac{1}{g(x+\Delta x) \cdot g(x)} \\
&= \left[\lim_{\Delta x \rightarrow 0} g(x) \cdot \lim_{\Delta x \rightarrow 0} \frac{[f(x+\Delta x) - f(x)]}{\Delta x} - \lim_{\Delta x \rightarrow 0} f(x) \cdot \lim_{\Delta x \rightarrow 0} \frac{[g(x+\Delta x) - g(x)]}{\Delta x} \right] \cdot \lim_{\Delta x \rightarrow 0} \left(\frac{1}{g(x+\Delta x) \cdot g(x)} \right) \\
&= [g(x) \cdot f'(x) - f(x) \cdot g'(x)] \cdot \left(\frac{1}{g(x) \cdot g(x)} \right) = \frac{g(x) \cdot f'(x) - f(x) \cdot g'(x)}{g(x) \cdot g(x)} = \frac{g(x) \cdot f'(x) - f(x) \cdot g'(x)}{[g(x)]^2}
\end{aligned}$$

Chain Rule for Derivative.

$$\text{Claim: } D_x (f(g(x))) = f'(g(x)) \cdot g'(x)$$

$$\text{Recall: Alternative form of derivative } f'(c) = \lim_{x \rightarrow c} \frac{[f(x) - f(c)]}{x - c};$$

$$\text{Hence, } f'(g(c)) = \lim_{x \rightarrow c} \left[\frac{f(g(x)) - f(g(c))}{g(x) - g(c)} \right]$$

Proof:

$$D_x (f(g(x))) = \lim_{x \rightarrow c} \left[\frac{f(g(x)) - f(g(c))}{x - c} \right] = \lim_{x \rightarrow c} \left[\frac{f(g(x)) - f(g(c))}{x - c} \cdot \frac{g(x) - g(c)}{g(x) - g(c)} \right] \quad \text{Assume } g(x) - g(c) \neq 0$$

$$= \lim_{x \rightarrow c} \left[\frac{f(g(x)) - f(g(c))}{g(x) - g(c)} \cdot \frac{g(x) - g(c)}{x - c} \right] = \lim_{x \rightarrow c} \left[\frac{f(g(x)) - f(g(c))}{g(x) - g(c)} \right] \cdot \lim_{x \rightarrow c} \left[\frac{g(x) - g(c)}{x - c} \right]$$

$$= f'(g(c)) \cdot g'(c)$$