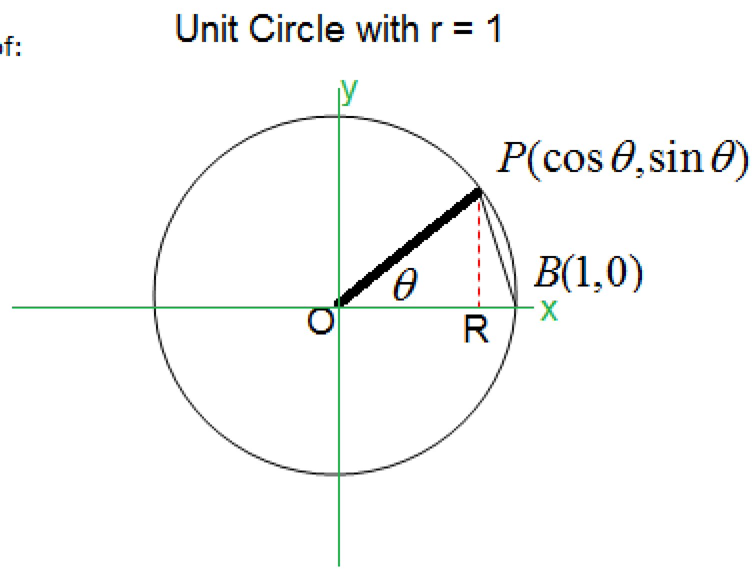
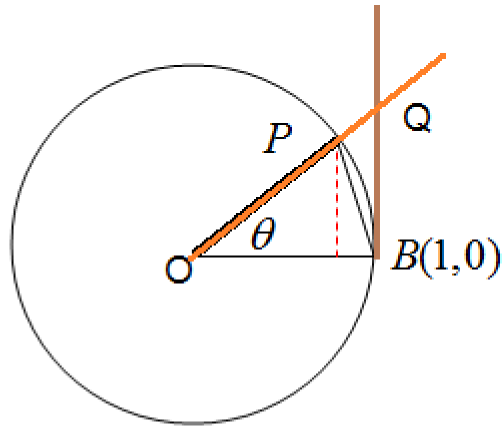
Claim: $\lim_{x \to 0} \frac{\sin x}{x} = 1$

Proof:





Note: $\sin\theta = \frac{\text{Length of PR}}{\text{Length of OP}} = \frac{\text{Length of PR}}{1} = \text{Length of PR}$

Note: $\tan\theta = \frac{\text{Length of BQ}}{\text{Length of OB}} = \frac{\text{Length of BQ}}{1} = \text{Length of BQ}$

Area of ΔOBP = $\frac{1}{2}(base)(height) = \frac{1}{2}(\text{length of OB})(\text{Length of PR}) = \frac{1}{2}(1)(\sin\theta) = \frac{1}{2}\sin\theta$

Area of Sector OBP = $\frac{1}{2}(\text{radius of circle})^2(\theta) = \frac{1}{2}(1)^2(\theta) = \frac{1}{2}\theta$

Area of ΔOBQ = $\frac{1}{2}(base)(height) = \frac{1}{2}(\text{length of OB})(\text{Length of BQ}) = \frac{1}{2}(1)(\tan\theta) = \frac{1}{2}\tan\theta$

Note: 0< Area of ΔOBP < Area of Sector OBP < Area of ΔOBQ

$$0 < \frac{1}{2}\sin\theta < \frac{1}{2}\theta < \frac{1}{2}\tan\theta$$

$$0 < \sin\theta < \theta < \tan\theta$$ Multiply each term by 2

$$0 < 1 < \frac{\theta}{\sin\theta} < \frac{\tan\theta}{\sin\theta}$$ Divide each term by $\sin\theta$;

$$0 < 1 < \frac{\theta}{\sin\theta} < \frac{1}{\cos\theta}$$ Note: $\frac{\tan\theta}{\sin\theta} = \frac{\sin\theta/\cos\theta}{\sin\theta} = \frac{1}{\cos\theta}$

$$1 > \frac{\sin\theta}{\theta} > \frac{\cos\theta}{1}$$ Taking reciprocals

$$1 > \frac{\sin\theta}{\theta} > \cos\theta$$

Hence, $\lim_{\theta\to 0} 1 > \lim_{\theta\to 0}\frac{\sin\theta}{\theta} > \lim_{\theta\to 0}\cos\theta$

$$1 > \lim_{\theta\to 0}\frac{\sin\theta}{\theta} > 1$$ Recall $\lim_{\theta\to 0}\cos\theta = 1.$

Therefore, $\lim_{\theta\to 0}\frac{\sin\theta}{\theta} = 1$ by The Squeeze (or Sandwich) Theorem.

Claim: $\lim_{x\to 0}\frac{1-\cos x}{x}=0$

Proof:

Note: $\sin^2 x+\cos^2 x=1 \Rightarrow \sin^2 x=1-\cos^2 x$

$$\frac{1-\cos x}{x}=\frac{1-\cos x}{x}\cdot\frac{1+\cos x}{1+\cos x}=\frac{1-\cos^2 x}{x\left(1+\cos x\right)}=\frac{\sin^2 x}{x\left(1+\cos x\right)}=\frac{\sin x\cdot\sin x}{x\left(1+\cos x\right)}=\frac{\sin x}{x}\cdot\frac{\sin x}{\left(1+\cos x\right)}$$

Hence, $\lim_{x\to 0}\frac{1-\cos x}{x}=\lim_{x\to 0}\left[\frac{\sin x}{x}\cdot\frac{\sin x}{\left(1+\cos x\right)}\right]=\lim_{x\to 0}\left[\frac{\sin x}{x}\right]\cdot\lim_{x\to 0}\left[\frac{\sin x}{\left(1+\cos x\right)}\right]=\left(1\right)\left(\frac{0}{1}\right)=0$

Claim: $\lim_{x\to 0}\frac{\cos x-1}{x}=0$

Proof:

$$\frac{\cos x-1}{x}=\frac{-1\left(\cos x-1\right)}{-1x}=\frac{1-\cos x}{-1x}=\frac{1}{-1}\cdot\frac{1-\cos x}{x}=\left(-1\right)\cdot\frac{1-\cos x}{x}$$

Hence, $\lim_{x\to 0}\frac{\cos x-1}{x}=\lim_{x\to 0}\left[\left(-1\right)\cdot\frac{1-\cos x}{x}\right]=\lim_{x\to 0}\left[\left(-1\right)\right]\cdot\lim_{x\to 0}\left[\frac{1-\cos x}{x}\right]=\left(-1\right)\left(0\right)=0$

Claim: $D_x\left(\sin x\right) = \cos x$

Recall: $\sin x\left(A + B\right) = \sin A \cdot \cos B + \cos A \cdot \sin B$

$$\lim_{x \to 0} \frac{\sin x}{x} = 1; \quad \lim_{x \to 0} \frac{1 - \cos x}{x} = 0; \quad \lim_{x \to 0} \frac{\cos x - 1}{x} = 0$$

Proof:

$$D_x\left(\sin x\right) = \lim_{\Delta x \to 0}\left[\frac{\sin\left(x + \Delta x\right) - \sin x}{\Delta x}\right] = \lim_{\Delta x \to 0}\left[\frac{\sin x \cdot \cos \Delta x + \cos x \cdot \sin \Delta x - \sin x}{\Delta x}\right]$$

$$= \lim_{\Delta x \to 0}\left[\frac{\cos x \cdot \sin \Delta x + \sin x \cdot \cos \Delta x - \sin x}{\Delta x}\right] = \lim_{\Delta x \to 0}\left[\frac{\cos x \cdot \sin \Delta x + \sin x \cdot \left(\cos \Delta x - 1\right)}{\Delta x}\right]$$

$$= \lim_{\Delta x \to 0}\left[\frac{\cos x \cdot \sin \Delta x}{\Delta x} + \frac{\sin x \cdot \left(\cos \Delta x - 1\right)}{\Delta x}\right] = \lim_{\Delta x \to 0}\left[\cos x \cdot \frac{\sin \Delta x}{\Delta x} + \sin x \cdot \frac{\left(\cos \Delta x - 1\right)}{\Delta x}\right]$$

$$= \lim_{\Delta x \to 0}\left[\cos x\right] \cdot \lim_{\Delta x \to 0}\left[\frac{\sin \Delta x}{\Delta x}\right] - \lim_{\Delta x \to 0}\left[\sin x\right] \cdot \lim_{\Delta x \to 0}\left[\frac{\left(\cos \Delta x - 1\right)}{\Delta x}\right]$$

$$= \cos x \cdot \left[1\right] - \sin x \cdot \left[0\right] = \cos x$$

Claim: $D_x\left(\cos x\right)=-\sin x$

Recall: $\cos x\left(A+B\right)=\cos A\cdot\cos B-\sin A\cdot\sin B$

$$\lim_{x\to 0}\frac{\sin x}{x}=1;\quad \lim_{x\to 0}\frac{1-\cos x}{x}=0;\quad \lim_{x\to 0}\frac{\cos x-1}{x}=0$$

Proof:

$$D_x\left(\cos x\right)=\lim_{\Delta x\to 0}\left[\frac{\cos\left(x+\Delta x\right)-\cos x}{\Delta x}\right]=\lim_{\Delta x\to 0}\left[\frac{\cos x\cdot\cos\Delta x-\sin x\cdot\sin\Delta x-\cos x}{\Delta x}\right]$$

$$=\lim_{\Delta x\to 0}\left[\frac{\cos x\cdot\cos\Delta x-\cos x-\sin x\cdot\sin\Delta x}{\Delta x}\right]=\lim_{\Delta x\to 0}\left[\frac{\cos x\cdot\left(\cos\Delta x-1\right)-\sin x\cdot\sin\Delta x}{\Delta x}\right]$$

$$=\lim_{\Delta x\to 0}\left[\frac{\cos x\cdot\left(\cos\Delta x-1\right)}{\Delta x}-\frac{\sin x\cdot\sin\Delta x}{\Delta x}+\right]=\lim_{\Delta x\to 0}\left[\cos x\cdot\frac{\left(\cos\Delta x-1\right)}{\Delta x}-\sin x\cdot\frac{\sin\Delta x}{\Delta x}\right]$$

$$=\lim_{\Delta x\to 0}\left[\cos x\right]\cdot\lim_{\Delta x\to 0}\left[\frac{\left(\cos\Delta x-1\right)}{\Delta x}\right]-\lim_{\Delta x\to 0}\left[\sin x\right]\cdot\lim_{\Delta x\to 0}\left[\frac{\sin\Delta x}{\Delta x}\right]$$

$$=\cos x\cdot\left[0\right]-\sin x\cdot\left[1\right]=-\sin x$$

Product Rule for Derivative.

Claim: $D_x\left(f(x)\cdot g(x)\right)=f(x)\cdot g'(x)+g(x)\cdot f'(x)$

Recall: $g'(x)=\lim\limits_{\Delta x\to 0}\dfrac{\left[g(x+\Delta x)-g(x)\right]}{\Delta x}$; $\qquad f'(x)=\lim\limits_{\Delta x\to 0}\dfrac{\left[f(x+\Delta x)-f(x)\right]}{\Delta x}$

Proof:

$$D_x\left(f(x)\cdot g(x)\right)=\lim_{\Delta x\to 0}\left[\frac{f(x+\Delta x)\cdot g(x+\Delta x)-f(x)\cdot g(x)}{\Delta x}\right]$$

$$=\lim_{\Delta x\to 0}\left[\frac{f(x+\Delta x)\cdot g(x+\Delta x)-f(x+\Delta x)\cdot g(x)+f(x+\Delta x)\cdot g(x)-f(x)\cdot g(x)}{\Delta x}\right]$$

$$=\lim_{\Delta x\to 0}\left[\frac{f(x+\Delta x)\cdot g(x+\Delta x)-f(x+\Delta x)\cdot g(x)}{\Delta x}+\frac{f(x+\Delta x)\cdot g(x)-f(x)\cdot g(x)}{\Delta x}\right]$$

$$=\lim_{\Delta x\to 0}\left[\frac{f(x+\Delta x)\cdot\left[g(x+\Delta x)-g(x)\right]}{\Delta x}+\frac{g(x)\cdot\left[f(x+\Delta x)-f(x)\right]}{\Delta x}\right]$$

$$=\lim_{\Delta x\to 0}\left[f(x+\Delta x)\cdot\frac{\left[g(x+\Delta x)-g(x)\right]}{\Delta x}+g(x)\cdot\frac{\left[f(x+\Delta x)-f(x)\right]}{\Delta x}\right]$$

$$=\lim_{\Delta x\to 0}f(x+\Delta x)\cdot\lim_{\Delta x\to 0}\frac{\left[g(x+\Delta x)-g(x)\right]}{\Delta x}+\lim_{\Delta x\to 0}g(x)\cdot\lim_{\Delta x\to 0}\frac{\left[f(x+\Delta x)-f(x)\right]}{\Delta x}$$

$$=f(x)\cdot g'(x)+g(x)\cdot f'(x)$$

Quotient Rule for Derivative.

Claim: $D_x\left(\frac{f(x)}{g(x)}\right)=\frac{g(x)\cdot f'(x)-f(x)\cdot g'(x)}{\left[g(x)\right]^2}, \qquad g(x)\neq 0$

Recall: $g'(x)=\lim_{\Delta x\to 0}\frac{\left[g(x+\Delta x)-g(x)\right]}{\Delta x}; \qquad f'(x)=\lim_{\Delta x\to 0}\frac{\left[f(x+\Delta x)-f(x)\right]}{\Delta x}; \qquad \frac{A}{B}-\frac{C}{D}=\frac{AD-BC}{DB}$

Proof:

$$D_x\left(\frac{f(x)}{g(x)}\right)=\lim_{\Delta x\to 0}\left[\frac{\frac{f(x+\Delta x)}{g(x+\Delta x)}-\frac{f(x)}{g(x)}}{\Delta x}\right]=\lim_{\Delta x\to 0}\left[\frac{\frac{f(x+\Delta x)\cdot g(x)-g(x+\Delta x)\cdot f(x)}{g(x+\Delta x)\cdot g(x)}}{\Delta x}\right]$$

$$=\lim_{\Delta x\to 0}\left[\frac{f(x+\Delta x)\cdot g(x)-g(x+\Delta x)\cdot f(x)}{g(x+\Delta x)\cdot g(x)}\cdot\frac{1}{\Delta x}\right]=\lim_{\Delta x\to 0}\left[\frac{f(x+\Delta x)\cdot g(x)-g(x+\Delta x)\cdot f(x)}{g(x+\Delta x)\cdot g(x)\cdot\Delta x}\right]$$

$$=\lim_{\Delta x\to 0}\left[\frac{f(x+\Delta x)\cdot g(x)-f(x)g(x)+f(x)g(x)-g(x+\Delta x)\cdot f(x)}{g(x+\Delta x)\cdot g(x)\cdot\Delta x}\right]$$

$$=\lim_{\Delta x\to 0}\left[\frac{g(x)\left[f(x+\Delta x)-f(x)\right]+f(x)\cdot\left[g(x)-g(x+\Delta x)\right]}{g(x+\Delta x)\cdot g(x)\cdot\Delta x}\right]$$

$$= \lim_{\Delta x \to 0} \left[\frac{g(x)\left[f(x+\Delta x) - f(x)\right] - f(x) \cdot \left[g(x+\Delta x) - g(x)\right]}{g(x+\Delta x) \cdot g(x) \cdot \Delta x} \right]$$

$$= \lim_{\Delta x \to 0} \left[\frac{g(x)\left[f(x+\Delta x) - f(x)\right] - f(x) \cdot \left[g(x+\Delta x) - g(x)\right]}{\Delta x} \right] \cdot \frac{1}{g(x+\Delta x) \cdot g(x)}$$

$$= \lim_{\Delta x \to 0} \left[\frac{g(x)\left[f(x+\Delta x) - f(x)\right]}{\Delta x} - \frac{f(x) \cdot \left[g(x+\Delta x) - g(x)\right]}{\Delta x} \right] \cdot \frac{1}{g(x+\Delta x) \cdot g(x)}$$

$$= \lim_{\Delta x \to 0} \left[g(x) \cdot \frac{\left[f(x+\Delta x) - f(x)\right]}{\Delta x} - \frac{f(x) \cdot \left[g(x+\Delta x) - g(x)\right]}{\Delta x} \right] \cdot \frac{1}{g(x+\Delta x) \cdot g(x)}$$

$$= \left[\lim_{\Delta x \to 0} g(x) \cdot \lim_{\Delta x \to 0} \frac{\left[f(x+\Delta x) - f(x)\right]}{\Delta x} - \lim_{\Delta x \to 0} f(x) \cdot \lim_{\Delta x \to 0} \frac{\left[g(x+\Delta x) - g(x)\right]}{\Delta x} \right] \cdot \lim_{\Delta x \to 0} \left(\frac{1}{g(x+\Delta x) \cdot g(x)} \right)$$

$$= \left[g(x) \cdot f'(x) - f(x) \cdot g'(x) \right] \cdot \left(\frac{1}{g(x) \cdot g(x)} \right) = \frac{g(x) \cdot f'(x) - f(x) \cdot g'(x)}{g(x) \cdot g(x)} = \frac{g(x) \cdot f'(x) - f(x) \cdot g'(x)}{\left[g(x)\right]^2}$$

Chain Rule for Derivative.

Claim: $D_x\left(f\left(g\left(x\right)\right)\right)=f'\left(g\left(x\right)\right)\cdot g'\left(x\right)$

Recall: Alternative form of derivative $\quad f'\left(c\right)=\lim\limits_{x\to c}\dfrac{\left[f\left(x\right)-f\left(c\right)\right]}{x-c}$;

$$\text{Hence, } f'\left(g\left(c\right)\right)=\lim_{x\to c}\left[\frac{f\left(g\left(x\right)\right)-f\left(g\left(c\right)\right)}{g\left(x\right)-g\left(c\right)}\right]$$

Proof:

$$D_x\left(f\left(g\left(c\right)\right)\right)=\lim_{x\to c}\left[\frac{f\left(g\left(c\right)\right)-f\left(g\left(c\right)\right)}{x-c}\right]=\lim_{x\to c}\left[\frac{f\left(g\left(c\right)\right)-f\left(g\left(c\right)\right)}{x-c}\cdot\frac{g\left(x\right)-g\left(c\right)}{g\left(x\right)-g\left(c\right)}\right] \quad \text{Assume } g\left(x\right)-g\left(c\right)\neq 0$$

$$=\lim_{x\to c}\left[\frac{f\left(g\left(c\right)\right)-f\left(g\left(c\right)\right)}{g\left(x\right)-g\left(c\right)}\cdot\frac{g\left(x\right)-g\left(c\right)}{x-c}\right]=\lim_{x\to c}\left[\frac{f\left(g\left(c\right)\right)-f\left(g\left(c\right)\right)}{g\left(x\right)-g\left(c\right)}\right]\cdot\lim_{x\to c}\left[\frac{g\left(x\right)-g\left(c\right)}{x-c}\right]$$

$$= f'\left(g\left(c\right)\right)\cdot g'\left(c\right)$$

Rolle's Theorem:

Claim: Let f be differentiable on $(a,\ b)$ and continuous on $[a,\ b]$.

If $f(a)=f(b)$, then there is at least one number $c\in(a,b)$ such that $f'(c)=0$.

Recall:

a) Extreme Value Theorem: If f is continuous on $[a,\ b]$, then f has both a minimum and maximum on $[a,\ b]$.

b) Relative Extrema: If f has a relative minimum or maximum at $x=c$, then c is a critical number.

Proof:

Let $f(a)=f(b)=k$.

Case 1: If $f(x)=k$ for all $x\in[a,b]$ then $f(x)$ is a constant function and $f'(x)=0$ for all $x\in(a,b)$.

Case 2: Suppose $f(x)>k$ for some $x\in(a,b)$. Then by Extreme Value Theorem f has maximum at some $c\in(a,b)$.

Find the arc length for $y = f(x) = \sqrt{x}$ with $x \in [a,b] = [0, 4]$.

Let L be the length of the arc.

First we divide the arc length into n segments.

The kth segment will have the following endpoints: $\left(x_{k-1}, f\left(x_{k-1}\right)\right)$ and $\left(x_k, f\left(x_k\right)\right)$.

Hence, length of kth segment = $\sqrt{\left(x_k - x_{k-1}\right)^2 + \left(f(x_k) - f(x_{k-1})\right)^2}$

Let $\Delta x_k = x_k - x_{k-1}$.

L_k = length of kth segment = $\sqrt{\left(\Delta x_k\right)^2 + \left(f(x_k) - f(x_{k-1})\right)^2}$

$$= \sqrt{\left(\Delta x_k\right)^2 \left[\frac{\left(\Delta x_k\right)^2}{\left(\Delta x_k\right)^2} + \frac{\left(f(x_k) - f(x_{k-1})\right)^2}{\left(\Delta x_k\right)^2}\right]} = \sqrt{\left[1 + \frac{\left(f(x_k) - f(x_{k-1})\right)^2}{\left(\Delta x_k\right)^2}\right]}\left(\Delta x_k\right)$$

By the Mean Value Theorem, there is $t_k \in \left(x_{k-1}, x_{k-1}\right)$ such that $\dfrac{f(x_k) - f(x_{k-1})}{x_k - x_{k-1}} = f'(t_k)$

Hence, $f(x_k) - f(x_{k-1}) = \left[x_k - x_{k-1}\right] f'(t_k) = \Delta x_k f'(t_k)$

$$L_k = \sqrt{\left[1 + \frac{\left(\Delta x_k f'(t_k)\right)^2}{\left(\Delta x_k\right)^2}\right]}\left(\Delta x_k\right) = \sqrt{\left[1 + \left[f'(t_k)\right]^2\right]}\left(\Delta x_k\right)$$

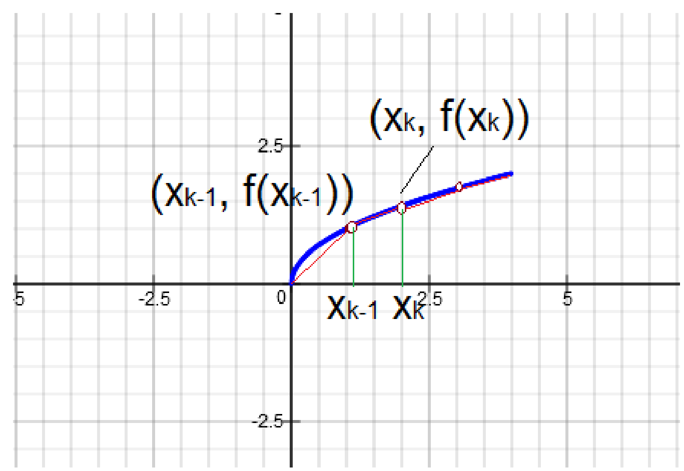
Therefore, $L \approx L_1 + L_2 + \ldots + L_n$

$$L \approx \sqrt{\left[1 + \left[f'(t_1)\right]^2\right]}\left(\Delta x_1\right) + \sqrt{\left[1 + \left[f'(t_2)\right]^2\right]}\left(\Delta x_1\right) + \ldots + \sqrt{\left[1 + \left[f'(t_n)\right]^2\right]}\left(\Delta x_n\right)$$

$$L \approx \sum_{i=1}^{n} \sqrt{\left[1 + \left[f'(t_i)\right]^2\right]}\left(\Delta x_i\right)$$

$$L = \lim_{\max \Delta x_k \to 0} \sum_{i=1}^{n} \sqrt{\left[1 + \left[f'(t_i)\right]^2\right]}\left(\Delta x_i\right)$$

$$L = \int_a^b \sqrt{1 + \left[f'(x)\right]^2}\, dx$$



Lenght of Arc

Arc Length of Plane Curve: $L = \int_a^b \sqrt{1 + [f'(x)]^2}\, dx = \int_a^b \sqrt{1 + \left[\frac{dy}{dx}\right]^2}\, dx =$

Arc Length of Parametric Curves: $L = \int_a^b \sqrt{\left[\frac{dx}{dt}\right]^2 + \left[\frac{dy}{dt}\right]^2}\, dx$

Arc Length of Polar Curve: $L = \int_\alpha^\beta \sqrt{r^2 + \left[\frac{dr}{d\theta}\right]^2}\, dx$