Infinite Series Tests

Common Infinite Series

a) $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if $p > 1$ and diverges if $0 < p \le 1$

b) $\sum_{n=1}^{\infty} \frac{\ln n}{n^p}$ converges if $p > 1$ and diverges if $0 < p \le 1$

Geometric Series Test for Infinite Series $\sum_{n=1}^{\infty} ar^n$

a) Geometric Series $\sum_{n=1}^{\infty} ar^n$ converges to $\frac{a}{1-r}$ if r is between -1 and 1.

b) Geometric Series $\sum_{n=1}^{\infty} ar^n$ diverges if r is not between -1 and 1.

*n*th Term Test for Infinite Series $\sum_{n=1}^{\infty} a_n$

a) If $\lim_{n\to\infty} a_n \ne 0,$ then the series $\sum_{n=1}^{\infty} a_n$ diverges.

b) If $\lim_{n\to\infty} a_n = 0,$ then the series $\sum_{n=1}^{\infty} a_n$ may converge (further investigation is needed).

Note: *n*th Term Test does NOT say that if $\lim_{n\to\infty} a_n = 0$ then the series $\sum_{n=1}^{\infty} a_n$ converges.

Examples:

For the series $\sum_{n=1}^{\infty} \frac{1}{n}$, $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{n} = 0$ and $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges to ∞.

For the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$, $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{n^2} = 0$ and $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges to a finite value.

p-Series Test for Infinite Series $\sum_{n=1}^{\infty} \frac{1}{n^p}$

a) If $p > 1$ then the series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges.

b) If $p \leq 1$ then the series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ diverges.

Alternating Series Test for Infinite Series $\sum_{n=1}^{\infty} (-1)^n a_n$

The alternating series $\sum_{n=1}^{\infty} (-1)^n a_n$ converges if the following conditions are satisfied:

a) $0 < a_{n+1} \leq a_n$

b) $\lim_{n \to \infty} a_n = 0$

Integral Test for Infinite Series $\sum_{n=1}^{\infty} a_n$ where $a_n \geq 0$

Let $f(n) = a_n$

a) The series $\sum_{n=1}^{\infty} a_n$ converges if $\int_1^{\infty} f(x)dx$ converges to a finite number.

b) The series $\sum_{n=1}^{\infty} a_n$ diverges if $\int_1^{\infty} f(x)dx$ diverges to infinity.

Basic Comparison Test for Infinite Series $\sum_{n=1}^{\infty} a_n$ where $a_n > 0$

a) To prove that $\sum_{n=1}^{\infty} a_n$ converges:

- Form a new series $\sum_{n=1}^{\infty} b_n$ such that $\sum_{n=1}^{\infty} a_n \leq \sum_{n=1}^{\infty} b_n$;
- Show that $\sum_{n=1}^{\infty} b_n$ converges.

b) To prove that $\sum_{n=1}^{\infty} a_n$ diverges:

- Form a new series $\sum_{n=1}^{\infty} b_n$ such that $\sum_{n=1}^{\infty} b_n \leq \sum_{n=1}^{\infty} a_n$;
- Show that $\sum_{n=1}^{\infty} b_n$ diverges to ∞.

Limit Comparison Test for Infinite Series $\sum_{n=1}^{\infty} a_n$ where $a_n > 0$

a) To prove that $\sum_{n=1}^{\infty} a_n$ converges:

- Form a new series $\sum_{n=1}^{\infty} b_n$ similar to $\sum_{n=1}^{\infty} a_n$;
- Show that $\lim_{n\to\infty} \frac{a_n}{b_n} = L > 0$;
- Show that $\sum_{n=1}^{\infty} b_n$ converges.

b) To prove that $\sum_{n=1}^{\infty} a_n$ diverges:

- Form a new series $\sum_{n=1}^{\infty} b_n$ similar to $\sum_{n=1}^{\infty} a_n$;
- Show that $\lim_{n\to\infty} \frac{a_n}{b_n} = L > 0$;
- Show that $\sum_{n=1}^{\infty} b_n$ diverges.

Absolute Convergence

The series $\sum_{n=1}^{\infty} a_n$ is said to converge absolutely if

$\sum_{n=1}^{\infty} a_n$ converges and $\sum_{n=1}^{\infty} |a_n|$ converges.

Conditional Convergence:

The series $\sum_{n=1}^{\infty} a_n$ is said to converge conditionally if

$\sum_{n=1}^{\infty} a_n$ converges and $\sum_{n=1}^{\infty} |a_n|$ diverges.

Ratio Test for Infinite Series $\sum_{n=1}^{\infty} a_n$

a) If $\lim_{x \to \infty} \left| \frac{a_{n+1}}{a_n} \right| < 1$, then the series $\sum_{n=1}^{\infty} a_n$ converges.

b) If $\lim_{x \to \infty} \left| \frac{a_{n+1}}{a_n} \right| > 1$, then the series $\sum_{n=1}^{\infty} a_n$ diverges.

c) If $\lim_{x \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = 1$, use another test.

Root Test for Infinite Series $\sum_{n=1}^{\infty} a_n$

a) If $\lim_{x \to \infty} \sqrt[n]{|a_n|} < 1$, then the series $\sum_{n=1}^{\infty} a_n$ converges.

b) If $\lim_{x \to \infty} \sqrt[n]{|a_n|} > 1$, then the series $\sum_{n=1}^{\infty} a_n$ diverges.

c) If $\lim_{x \to \infty} \sqrt[n]{|a_n|} = 1$, use another test.

n*th* Degree Taylor Polynomial $P_n(x)$

The polynomial $P_n(x)$ can be used to approximate values for $f(x)$ for values around c.

$$P_n(x) = f(c) + f'(c)(x-c) + \frac{f''(c)(x-c)^2}{2} + \frac{f'''(c)(x-c)^3}{3!}$$
$$+ \frac{f^{(4)}(c)(x-c)^4}{4!} + \ldots + \frac{f^{(n)}(c)(x-c)^n}{n!}$$

Taylor Series $\sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(x-c)^n$ Centered at c

$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(x-c)^n$ is a power series representation of $f(x)$.

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(x-c)^n$$
$$= \frac{f(c)}{0!}(x-c)^0 + \frac{f'(c)}{1!}(x-c)^2 + \frac{f''(c)}{2!}(x-c)^2$$
$$+ \frac{f'''(c)}{3!}(x-c)^3 + \frac{f^{(4)}(c)}{4!}(x-c)^4 + \ldots$$
$$+ \frac{f^{(n)}(c)}{n!}(x-c)^n + \ldots$$