Test 1 Review

1) $P(1,2,3)$ $\quad Q(4,2,7)$ $\quad R(1,0,1)$

a) Find $\underset{\sim}{u} = \overrightarrow{PQ} = \langle 4-1,\ 2-2,\ 7-3 \rangle = \langle 3,0,4 \rangle$

$\underset{\sim}{v} = \overrightarrow{PR} = \langle 1-1,\ 0-2,\ 1-3 \rangle = \langle 0,-2,-2 \rangle$

b) $3\underset{\sim}{u} + 3\underset{\sim}{v} = 3\cdot\langle 3,0,4 \rangle + 3\cdot\langle 0,-2,-2 \rangle$

$= \langle 9, 0, 12 \rangle + \langle 0, -6, -6 \rangle$

$= \langle 9, -6, 6 \rangle$

(2) Find the ditance between $P(1, 4, 2)$ and $Q(-2, 4, 7)$.

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

$$d = \sqrt{(-2-1)^2 + (4-4)^2 + (7-2)^2} = \sqrt{34}$$

(3) Find equation of sphere with center at $(3, -1, 5)$ and radius of 5.

$$(x - x_1)^2 + (y - y_1)^2 + (z - z_1)^2 = r^2$$

$$(x-3)^2 + (y - -1)^2 + (z - 5)^2 = 5^2$$

$$(x-3)^2 + (y+1)^2 + (z-5)^2 = 25$$

(4) $x^2 + y^2 + z^2 - 4x - 6y - 2z = 2$

Find center and radius of sphere.

$$(x^2 - 4x + 4) + (y^2 - 6y + 9) + (z^2 - 2z + 1) = 2 \;\; +4 \;\; +9 \;\; +1$$

$-2 + -2 = -4$

$(-2)\cdot(-2) = 4$

$(-3) + (-3) = -6$

$(-3)(-3) = 9$

$-1 + -1 = -2$

$(-1)(-1) = 1$

$$(x-2)(x-2) + (y-3)(y-3) + (z-1)(z-1) = 16$$

$$(x-2)^2 + (y-3)^2 + (z-1)^2 = 16$$

Center of sphere $= (2, 3, 1)$

$r^2 = 16$

$r = \text{radius} = 4$

(5) Let $P(4, 2, 1)$ $\quad Q(3, 5, 7)$ $\quad R(1, 2, 3)$

Let $\underset{\sim}{u} = \overrightarrow{PQ} = \langle -1, 3, 6 \rangle$

$\underset{\sim}{v} = \overrightarrow{PR} = \langle -3, 0, 2 \rangle$

Find $\underset{\sim}{u} \cdot \underset{\sim}{v}$, $\underset{\sim}{v} \cdot \underset{\sim}{v}$, $\|\underset{\sim}{v}\|^2$

$$\underset{\sim}{u} \cdot \underset{\sim}{v} = \langle -1, 3, 6 \rangle \cdot \langle -3, 0, 2 \rangle$$
$$= 3 + 0 + 12 = 15$$

$$\underset{\sim}{v} \cdot \underset{\sim}{v} = \langle -3, 0, 2 \rangle \cdot \langle -3, 0, 2 \rangle = 9 + 0 + 4 = 13$$

$$\|\underset{\sim}{v}\| = \sqrt{(-3)^2 + 0^2 + 2^2} = \sqrt{13}$$

$$\|\underset{\sim}{v}\|^2 = (\sqrt{13})^2 = 13$$

(6) Let $\underset{\sim}{u} = \langle 10, -2, 4 \rangle$; $\underset{\sim}{v} = \langle -3, 4, 2 \rangle$

Find the angle θ between $\underset{\sim}{u}$ and $\underset{\sim}{v}$.

$$\cos\theta = \frac{\underset{\sim}{u} \cdot \underset{\sim}{v}}{\|\underset{\sim}{u}\| \cdot \|\underset{\sim}{v}\|}$$

$$\|\underset{\sim}{u}\| = \sqrt{(10)^2 + (-2)^2 + (4)^2} = \sqrt{120}$$

$$\|\underset{\sim}{v}\| = \sqrt{(-3)^2 + 4^2 + 2^2} = \sqrt{29}$$

$$\underset{\sim}{u} \cdot \underset{\sim}{v} = \langle 10, -2, 4 \rangle \cdot \langle -3, 4, 2 \rangle = -30 + -8 + 8 = -30$$

$$\cos\theta = \frac{-30}{\sqrt{120} \cdot \sqrt{29}} = -0.508547$$

$$\theta = \cos^{-1}(-0.508547) = 2.104 \text{ rad.}$$

(7) Let $\underset{\sim}{u} = \langle 7, -1, 2 \rangle$; $\underset{\sim}{v} = \langle 4, 2, 5 \rangle$

Are $\underset{\sim}{u}$ and $\underset{\sim}{v}$ orthogonal, parallel, or neither?

$$\underset{\sim}{u} \cdot \underset{\sim}{v} = \langle 7, -1, 2 \rangle \cdot \langle 4, 2, 5 \rangle = 28 + {}^{-}2 + 10 = 36$$

so $\underset{\sim}{u}$ and $\underset{\sim}{v}$ are NOT orthogonal.

Note : $\frac{7}{4}$, $\frac{-1}{2}$, $\frac{2}{5}$

Since these ratios are not the same, $\underset{\sim}{u}$ and $\underset{\sim}{v}$ are not parallel.

(8) Let $\underset{\sim}{u} = \langle 1, 2, -1 \rangle$; $\underset{\sim}{v} = \langle 2, 4, 3 \rangle$

Find project of $\underset{\sim}{u}$ onto $\underset{\sim}{v}$ = $\text{Proj}_{\underset{\sim}{v}} \underset{\sim}{u} = \left(\dfrac{\underset{\sim}{u} \cdot \underset{\sim}{v}}{\|\underset{\sim}{v}\|^2} \right) \cdot \underset{\sim}{v}$

$\underset{\sim}{u} \cdot \underset{\sim}{v} = \langle 1, 2, -1 \rangle \cdot \langle 2, 4, 3 \rangle = 2 + 8 + {}^{-}3 = 7$

$\|\underset{\sim}{v}\| = \sqrt{(2)^2 + 4^2 + 3^2} = \sqrt{29}$

$\|\underset{\sim}{v}\|^2 = (\sqrt{29})^2 = 29$

$\text{Proj}_{\underset{\sim}{v}} \underset{\sim}{u} = \left(\dfrac{7}{29} \right) \cdot \langle 2, 4, 3 \rangle = \langle 14/29, \dfrac{28}{29}, \dfrac{21}{29} \rangle$

(9) Let $\underset{\sim}{u} = \langle 1, 2, -1 \rangle$; $\underset{\sim}{v} = \langle 2, 4, 3 \rangle$

Find $\underset{\sim}{v} \times \underset{\sim}{u}$.

$$\underset{\sim}{v} \times \underset{\sim}{u} = \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underline{k} \\ 2 & 4 & 3 \\ 1 & 2 & -1 \end{vmatrix}$$

$$= (-4-6)\underset{\sim}{i} - (-2-3)\underset{\sim}{j} + (4-4)\underline{k}$$

$$= -10\underset{\sim}{i} + 5\underset{\sim}{j} + 0\underline{k} = -10\underset{\sim}{i} + 5\underset{\sim}{j}$$

$$= \langle -10, 5, 0 \rangle$$

(10) Let $P(3, 0, 4)$; $Q(1, 2, 5)$

Let L be the line passing through P and Q.

Find set of parametric equations for L.

Let $\underline{v}$ = direction vector = $\overrightarrow{PQ} = \langle -2, 2, 1 \rangle$

Equations for L:

$$\begin{aligned} x - x_1 &= at \\ y - y_1 &= bt \\ z - z_1 &= ct \end{aligned} \quad \Rightarrow \quad \begin{aligned} x - 3 &= -2t \\ y - 0 &= 2t \\ z - 4 &= 1t \end{aligned}$$

(11) Let $P(3,0,4)$ $Q(1,2,5)$ $R(1,5,4)$

Find the plane containing P, Q, and R.

Let $\underset{\sim}{u} = \overrightarrow{PQ} = \langle -2, 2, 1 \rangle$

$\underset{\sim}{v} = \overrightarrow{PR} = \langle -2, 5, 0 \rangle$

So $\underset{\sim}{u}$ and $\underset{\sim}{v}$ are vectors on the plane.

Find vector normal($\underset{\sim}{n}$) to the plane:

$$\text{Let } \underset{\sim}{n} = \underset{\sim}{u} \times \underset{\sim}{v} = \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ -2 & 2 & 1 \\ -2 & 5 & 0 \end{vmatrix}$$

$$= (0-5)\underset{\sim}{i} - (0 - -2)\underset{\sim}{j} + (-10 - -4)\underset{\sim}{k}$$

$$= \langle -5, -2, -6 \rangle$$

So equation of plane containing P, Q, and R:

$$a(x - x_1) + b(y - y_1) + c(z - z_1) = 0$$

$$-5(x - 3) - 2(y - 0) - 6(z - 4) = 0$$

(12) Let $(x, y, z) = (4, 2, 1)$ Point in Rectangular system

Find corresponding points in the cylindrical and spherical systems.

Note: $x^2 + y^2 = r^2$

$16 + 4 = r^2$

$r = \sqrt{20}$

$\tan\theta = y/x$

$\tan\theta = 2/4$

$\theta = \tan^{-1}(1/2) = 0.463647$

Cylindrical $(r, \theta, z) = (\sqrt{20}, 0.463647, 1)$

$$\rho^2 = x^2 + y^2 + z^2 = 16 + 4 + 1 = 21$$

$$\rho = \sqrt{21}$$

$$\theta = 0.463647$$

$$\phi = \cos^{-1}\left(\frac{z}{\sqrt{x^2+y^2+z^2}}\right) = \cos^{-1}\left(\frac{1}{\sqrt{21}}\right) = 1.350808 \text{ rad.}$$

Spherical: $(\rho, \theta, \phi) = (\sqrt{21},\ 0.463647 \text{ rad},\ 1.350808 \text{ rad})$

(13) $x^2 + y^2 = 4z$ Equ. in Rectangular System

Find corresponding equations in the cylindrical and spherical systems.

Note: $x^2 + y^2 = r^2$

So, $x^2 + y^2 = 4z$

$r^2 = 4z$

$r = \pm\sqrt{4z}$ Equ. in Cylindrical System.

Note: $x = \rho \sin\phi \cos\theta$

$y = \rho \sin\phi \sin\theta$

$z = \rho \cos\phi$

So $x^2 + y^2 = \cancel{4}4z$

$(\rho \sin\phi \cos\theta)^2 + (\rho \sin\phi \sin\theta)^2$

$= 4\rho\cos\phi$

Corresponding equ. in the Spherical system

(4) Let $r = 2\cos\theta$ be an equ. in the cylindrical system.

Find the corresponding equations in the ~~cylindrical~~ rectangular and spherical systems.

So $r = 2\cos\theta$

$r \cdot r = r \cdot 2\cos\theta$

$r^2 = 2 \cdot r\cos\theta$

$x^2 + y^2 = 2 \cdot x$ Equ. in Rectangular System.

Note: $x = \rho\sin\phi\cos\theta$, $y = \rho\sin\phi\sin\theta$, $z = \rho\cos\phi$

$(\rho\sin\phi\cos\theta)^2 + (\rho\sin\phi\sin\theta)^2 = 2 \cdot \rho\sin\phi\cos\theta$

Equ. in Spherical System

(15) Let $\rho = 5\cos\phi$ be an equ. in the spherical system.

Find the corresponding equs. in the rectangular and Cylindrical systems.

So
$$\rho = 5\cos\phi$$
$$\rho\cdot\rho = 5\cdot\rho\cos\phi$$
$$\rho^2 = 5\rho\cos\phi$$
$$x^2 + y^2 + z^2 = 5\cdot z \quad \text{Equ. in Rectangular System.}$$

~~So~~ $\cancel{((\rho\sin(\phi)\cos(\theta))^2 + ((\rho\sin\phi\sin(\theta))^2 + ((\rho\cos(\phi))^2}$

So $$r^2 + z^2 = 5z \quad \text{Equ. in Cylindrical System.}$$

(16) Let $\underset{\sim}{r}(t) = \langle t, t^2, 1/t \rangle$

$\underset{\sim}{r}(0) = \langle 0, 0, \text{undefined} \rangle$

$\underset{\sim}{r}(\Delta t) = \langle \Delta t, (\Delta t)^2, 1/\Delta t \rangle$

Domain of $\underset{\sim}{r}(t) = \{$All real numbers, except $0\}$

Corresponding Parametric Equs:

$$x = t$$
$$y = t^2$$
$$z = 1/t$$

(17) Let $P(1, 2, 3)$; $Q(4, 5, 9)$

Find line L passing through P and Q.

Find corresponding vector-valued ~~#~~ $\underset{\sim}{r}(t)$ for line L.

Let $\underset{\sim}{v} = \overrightarrow{PQ} = \langle 3, 3, 6 \rangle$ = direction vector

Line L:
$$\begin{aligned} x - x_1 &= at \\ y - y_1 &= bt \\ z - z_1 &= ct \end{aligned} \quad \text{or} \quad \begin{aligned} x - 1 &= 3t \\ y - 2 &= 3t \\ z - 3 &= 6t \end{aligned} \quad \text{or} \quad \begin{aligned} x &= 3t+1 \\ y &= 3t+2 \\ z &= 6t+3 \end{aligned}$$

Corresponding $\underset{\sim}{r}(t)$:

$$\underset{\sim}{r}(t) = \langle 3t+1,\ 3t+2,\ 6t+3 \rangle$$

(18) Let $3x+5y-15=0$ be an equ. of a line in the rectangular system.

Find corresponding parametric equations and vector-valued function $\underset{\sim}{r}(t)$.

Note:

~~tet~~ $3x+5y-15=0$

$$5y = -3x + 15$$

$$y = -\frac{3}{5}x + 3$$

Let ~~t~~ $x = t$

So $y = -\frac{3}{5}t + 3$

$\Bigg]$ Parametric Equs.

So $\underset{\sim}{r}(t) = \langle t, -\frac{3}{5}t + 3 \rangle$ $\Bigg]$ Vector-valued function

(19) Find $\lim_{t \to 1} \left(t\underset{\sim}{i} + \sqrt{2-t}\,\underset{\sim}{j} + 4\underset{\sim}{k} \right)$

$$= 1\underset{\sim}{i} + 1\underset{\sim}{j} + 4\underset{\sim}{k} = \langle 1, 1, 4 \rangle$$

(20) $\underset{\sim}{r}(t) = \langle 5\cos t, \ 4\sin t, \ t \rangle$

$\underset{\sim}{r}'(t) = \langle -5\sin t, \ 4\cos t, \ 1 \rangle$

$\underset{\sim}{r}''(t) = \langle -5\cos t, \ -4\sin t, \ 0 \rangle$

$$\underset{\sim}{r}'(t) \cdot \underset{\sim}{r}''(t) = 25 \sin t \cos t \cancel{+} -16 \sin t \cos t + 0$$

$$= 9 \sin t \cos t$$

$$\underset{\sim}{r}'(t) \times \underset{\sim}{r}''(t) = \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ -5\sin t & 4\cos t & 1 \\ -5\cos t & -4\sin t & 0 \end{vmatrix}$$

$$= (0 - -4\sin t)\,\hat{i} - (0 - {}^{-}5\cos t)\,\underset{\sim}{j}$$

$$+ (20\sin^2 t + 20\cos^2 t)\,\underset{\sim}{k}$$

$$= \langle 4\sin t, -5\cos t, 20 \rangle$$

(21) Find $\int (1\underset{\sim}{i} + 4t\underset{\sim}{j} + t^2\underset{\sim}{k})\,dt$

$$= \left\langle t + C_1, \frac{4t^2}{2} + C_2 + \frac{t^3}{3} + C_3 \right\rangle$$

$$= \langle t + C_1, 2t^2 + C_2 + t^3/3 + C_3 \rangle$$

(22) Find $\int \langle \sin t\, \underset{\sim}{i} + 4\cos t\, \underset{\sim}{j} + t^3 \underset{\sim}{k} \rangle \, dt$

$$= \langle -\cos t + C_1,\ 4\sin t + C_2,\ t^4/4 + C_3 \rangle$$

(23) Find $\underset{\sim}{r}(t)$ that satisfies the initial condition $\underset{\sim}{r}(0) = \langle 1, 0, 2 \rangle$.

Given: $\underset{\sim}{r}'(t) = \langle 2t, e^t, t^2 \rangle$

So $\underset{\sim}{r}(t) = \int \underset{\sim}{r}'(t)\, dt$

$$= \int \langle 2t, e^t, t^2 \rangle \, dt$$

$$= \langle 2t^2/2 + C_1,\ e^t + C_2,\ t^3/3 + C_3 \rangle$$

Find a_T = Tangential Component of Acceleration

$$a_T = \underset{\sim}{a} \cdot \underset{\sim}{T} = \frac{\underset{\sim}{r}'(t) \cdot \underset{\sim}{r}''(t)}{\| \underset{\sim}{r}'(t) \|}$$

Let $t = 1$:

$$a_T = \frac{\langle 2, 2, 3 \rangle \cdot \langle 0, 2, 6 \rangle}{\| \langle 2, 2, 3 \rangle \|}$$

$$= \frac{0 + 4 + 18}{\sqrt{4 + 4 + 9}} = \frac{22}{\sqrt{17}} = 5.335783$$

a_N = Normal component of Acceleration

$$a_N = \sqrt{\| \underset{\sim}{a} \|^2 - (a_T)^2}$$

$$= \sqrt{40 - (5.335783)^2} = \sqrt{11.529419}$$

$$= 3.3955$$

So $\mathbf{r}(0) = \langle 0 + c_1 , e^0 + c_2 , 0 + c_3 \rangle$

$\langle 1, 0, 2 \rangle = \langle c_1 , 1 + c_2 + c_3 \rangle$

$\Longrightarrow \quad c_1 = 1, \quad 1 + c_2 = 0, \quad c_3 = 2$

$\qquad\qquad\qquad\qquad c_2 = -1$

So $\mathbf{r}(t) = \langle t^2 + 1, e^t - 1, t^3/3 + 2 \rangle$

(24) Let $\mathbf{r}(t) = \langle 2t, t^2, t^3 \rangle$

$\mathbf{v}(t) = \mathbf{r}'(t) = \langle 2, 2t, 3t^2 \rangle$

$\mathbf{a}(t) = \mathbf{r}''(t) = \langle 0, 2, 6t \rangle$

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|} = \frac{\langle 2, 2t, 3t^2 \rangle}{\sqrt{4 + 4t^2 + 9t^4}}$$

Find $K =$ Curvature at $t = 1$.

$$= \frac{\| \underset{\sim}{r}'(1) \times \underset{\sim}{r}''(1) \|}{\| \underset{\sim}{r}(1) \|^3}$$

$$\underset{\sim}{r}'(1) \times \underset{\sim}{r}''(1) = \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ 2 & 2 & 3 \\ 0 & 2 & 6 \end{vmatrix}$$

$$= \langle 6, -12, 4 \rangle$$

$$\| \underset{\sim}{r}'(1) \times \underset{\sim}{r}''(1) \| = \sqrt{36 + 144 + 16} = \sqrt{196} = 14$$

$$\| \underset{\sim}{r}'(1) \| = \sqrt{4 + 4 + 9} = \sqrt{17}$$

$$K = \frac{\sqrt{196}}{\left(\sqrt{17}\right)^3} = \frac{14}{70.0927} = 0.19973$$