Test 2 Review

1) Let $a_n = \dfrac{3n+2}{n}$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty}\left(\frac{3n+2}{n}\right) = \lim_{n\to\infty}\left(\frac{3}{1}\right) = 3$$

2) $\mathrm{a}_n = \sin\dfrac{n\pi}{2}$ Find $\lim\limits_{n\to\infty} \mathrm{a}_n$

For $n=1$, $\mathrm{a}_1 = \sin\dfrac{1\pi}{2} = 1$

For $n=2$, $\mathrm{a}_2 = \sin\dfrac{2\pi}{2} = 0$

For $n=3$, $\mathrm{a}_3 = \sin\dfrac{3\pi}{2} = -1$

For $n=4$, $\mathrm{a}_4 = \sin\dfrac{4\pi}{2} = 0$

Hence sequence of $\{\mathrm{a}_n\} = \{1,0,-1,0,1,0,-1,0,1,0,-1,0,1,0,-1,0,1,0,-1,0,1,0,-1,0,....\}$

Therefore $\lim\limits_{n\to\infty} \mathrm{a}_n$ diverges.

3) Let $\mathrm{a}_n = \left(\dfrac{2}{3}\right)^n + 5$

$$\lim_{n\to\infty} \mathrm{a}_n = \lim_{n\to\infty}\left(\left(\frac{2}{3}\right)^n + 5\right) = \lim_{n\to\infty}\left(\frac{2}{3}\right)^n + \lim_{n\to\infty}(5) = 0+5=5$$

4) Let $\mathrm{a}_n = \dfrac{5n}{n^2+1}$

$$\lim_{n\to\infty} \mathrm{a}_n = \lim_{n\to\infty}\left(\frac{5n}{n^2+1}\right) = \lim_{n\to\infty}\left(\frac{5}{2n}\right) = \frac{5}{\infty} = 0$$

5) $\sum_{n=0}^{\infty}\left(\frac{2}{7}\right)^n$. Explain why this series converges.

Explanation:

$\sum_{n=0}^{\infty}\left(\frac{2}{7}\right)^n$ is a Geometric Series with $r=\frac{2}{7}$.

By Geometric Series Theorem $\sum_{n=0}^{\infty}\left(\frac{2}{7}\right)^n$ converges because -1 < r < 1.

Also, $\sum_{n=0}^{\infty}\left(\frac{2}{7}\right)^n=\frac{a}{1-r}=\frac{1}{1-\frac{2}{7}}=\frac{7}{5}$

6) $\sum_{n=0}^{\infty}\left[(0.4)^n+(0.11)^n\right]$. Explain why this series converges.

Explanation:

$$\sum_{n=0}^{\infty}\left[(0.4)^n+(0.11)^n\right]=\sum_{n=0}^{\infty}\left[(0.4)^n\right] + \sum_{n=0}^{\infty}\left[+(0.11)^n\right]$$

$\sum_{n=0}^{\infty}\left[(0.4)^n\right]$ is a Geometric Series with $r=0.4$.

Hence $\sum_{n=0}^{\infty}\left[(0.4)^n\right]$ converges by Geometric Series Theorem.

$\sum_{n=0}^{\infty}\left[(0.11)^n\right]$ is a Geometric Series with $r=0.11$.

Hence $\sum_{n=0}^{\infty}\left[(0.11)^n\right]$ converges by Geometric Series Theorem

Therefore, $\sum_{n=0}^{\infty}\left[(0.4)^n+(0.11)^n\right]=\sum_{n=0}^{\infty}\left[(0.4)^n\right] + \sum_{n=0}^{\infty}\left[+(0.11)^n\right]$ converges.

7) $\sum_{n=0}^{\infty}(1.7)^n$.   Explain why this series diverges.

Explanation:

$\sum_{n=0}^{\infty}(1.77)^n$ is a Geometric Series with $r = 1.7$.

By Geometric Series Theorem $\sum_{n=0}^{\infty}(1.7)^n$ diverges because r > 1.

8) $\sum_{n=1}^{\infty}\frac{n+10}{10n+1}$ Use the *n*th Term Test to explain why the series $\sum_{n=1}^{\infty}\frac{n+10}{10n+1}$ diverges.

Explanation:

Let $a_n = \dfrac{n+10}{10n+1}$ and $\lim_{n\to\infty} a_n = \lim_{n\to\infty}\dfrac{n+10}{10n+1} = 1/10$

*n*th Term Test says that if $\lim_{n\to\infty} a_n \neq 0$ then series diverges.

9) $\sum_{n=2}^{\infty}(-1)^n\frac{n}{\ln n}$.   Explain why this series diverges. Use nth Term Test.

Explanation:

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty}\frac{n}{\ln n} = \lim_{n\to\infty}\left(\frac{1}{1/n}\right) = \lim_{n\to\infty}(n) = \infty$$

By nth Term Test $\sum_{n=2}^{\infty}(-1)^n\frac{n}{\ln n}$ diverges because $\lim_{n\to\infty} a_n \neq 0$.

10) $\sum_{n=0}^{\infty} \frac{2n+1}{5n+2}$. Explain why this series diverges. Use nth Term Test.

Explanation:

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{2n+1}{5n+2} = \lim_{n\to\infty}\left(\frac{2}{5}\right) = \frac{2}{5}$$

By nth Term Test $\sum_{n=0}^{\infty} \frac{2n+1}{5n+2}$ diverges because $\lim_{n\to\infty} a_n \neq 0$.

11) Telescoping Series $\sum_{n=1}^{\infty}\left(\frac{1}{n+2} - \frac{1}{n+3}\right)$. Does the series converge or diverge?

Note: $\lim_{n\to\infty} a_n = \lim_{n\to\infty}\left(\frac{1}{n+2} - \frac{1}{n+3}\right) = 0$

$Note:\ a_1 = \frac{1}{3} - \frac{1}{4};\ a_2 = \frac{1}{4} - \frac{1}{5};\ a_3 = \frac{1}{5} - \frac{1}{6}$

a) $S_2 = \left(\frac{1}{3} - \frac{1}{4}\right) + \left(\frac{1}{4} - \frac{1}{5}\right) = \frac{1}{3} - \frac{1}{5}$

b) $S_3 = \left(\frac{1}{3} - \frac{1}{4}\right) + \left(\frac{1}{4} - \frac{1}{5}\right) + \left(\frac{1}{5} - \frac{1}{6}\right) = \frac{1}{3} - \frac{1}{6}$

c) $S_4 = \left(\frac{1}{3} - \frac{1}{4}\right) + \left(\frac{1}{4} - \frac{1}{5}\right) + \left(\frac{1}{5} - \frac{1}{6}\right) + \left(\frac{1}{6} - \frac{1}{7}\right) = \frac{1}{3} - \frac{1}{7}$

d) $S_{10} = \left(\frac{1}{3} - \frac{1}{4}\right) + \left(\frac{1}{4} - \frac{1}{5}\right) + \left(\frac{1}{5} - \frac{1}{6}\right) + \left(\frac{1}{6} - \frac{1}{7}\right) + \left(\frac{1}{7} - \frac{1}{8}\right) + \left(\frac{1}{8} - \frac{1}{9}\right) + \left(\frac{1}{9} - \frac{1}{10}\right) + \left(\frac{1}{10} - \frac{1}{11}\right) + \left(\frac{1}{11} - \frac{1}{12}\right) + \left(\frac{1}{12} - \frac{1}{13}\right)$

$= \frac{1}{3} - \frac{1}{13}$

e) $S_k = \left(\frac{1}{3} - \frac{1}{4}\right) + \left(\frac{1}{4} - \frac{1}{5}\right) + \left(\frac{1}{5} - \frac{1}{6}\right) + \left(\frac{1}{6} - \frac{1}{7}\right) + \left(\frac{1}{7} - \frac{1}{8}\right) + \cdots + \left(\frac{1}{(k-1)+2} - \frac{1}{(k-1)+3}\right) + \left(\frac{1}{k+2} - \frac{1}{k+3}\right)$

$S_k = \frac{1}{3} - \frac{1}{k+3}$

f) $\sum_{n=1}^{\infty}\left(\frac{1}{n+2} - \frac{1}{n+3}\right) = \lim_{k\to\infty} S_k = \left(\frac{1}{3} - \frac{1}{k+3}\right) = \frac{1}{3} - 0 = \frac{1}{3}$

g) Telescoping Series $\sum_{n=1}^{\infty}\left(\frac{1}{n+2} - \frac{1}{n+3}\right)$ converges to $\frac{1}{3}$.

12) Telescoping Series $\sum_{n=1}^{\infty}\frac{1}{n^2+3n+2}$. Does the series converge or diverge?

Note: $\lim_{n\to\infty} a_n = \lim_{n\to\infty}\frac{1}{n^2+3n+2} = 0$

Decomposing $\frac{1}{n^2+3n+2}$: $\frac{1}{n^2+3n+2} = \frac{1}{(n+1)(n+2)} = \frac{A}{n+1} + \frac{B}{n+2} = \frac{1}{n+1} - \frac{1}{n+2}$

Note: $a_1 = \frac{1}{2}-\frac{1}{3}$; $a_2 = \frac{1}{3}-\frac{1}{4}$; $a_3 = \frac{1}{4}-\frac{1}{5}$

a) $S_2 = \left(\frac{1}{2}-\frac{1}{3}\right)+\left(\frac{1}{3}-\frac{1}{4}\right) = \frac{1}{2}-\frac{1}{4}$

b) $S_3 = \left(\frac{1}{2}-\frac{1}{3}\right)+\left(\frac{1}{3}-\frac{1}{4}\right)+\left(\frac{1}{4}-\frac{1}{5}\right) = \frac{1}{2}-\frac{1}{5}$

c) $S_4 = \left(\frac{1}{2}-\frac{1}{3}\right)+\left(\frac{1}{3}-\frac{1}{4}\right)+\left(\frac{1}{4}-\frac{1}{5}\right)+\left(\frac{1}{5}-\frac{1}{6}\right) = \frac{1}{2}-\frac{1}{6}$

d) $S_{10} = \left(\frac{1}{2}-\frac{1}{3}\right)+\left(\frac{1}{3}-\frac{1}{4}\right)+\left(\frac{1}{4}-\frac{1}{5}\right)+\left(\frac{1}{5}-\frac{1}{6}\right)+\left(\frac{1}{6}-\frac{1}{7}\right)+\left(\frac{1}{7}-\frac{1}{8}\right)+\left(\frac{1}{8}-\frac{1}{9}\right)+\left(\frac{1}{9}-\frac{1}{10}\right)+\left(\frac{1}{10}-\frac{1}{11}\right)+\left(\frac{1}{11}-\frac{1}{12}\right)$

$= \frac{1}{2}-\frac{1}{12}$

e) $S_k = \left(\frac{1}{2}-\frac{1}{3}\right)+\left(\frac{1}{3}-\frac{1}{4}\right)+\left(\frac{1}{4}-\frac{1}{5}\right)+\left(\frac{1}{5}-\frac{1}{6}\right)+\left(\frac{1}{6}-\frac{1}{7}\right)+\cdots+\left(\frac{1}{(k-1)+1}-\frac{1}{(k-1)+2}\right)+\left(\frac{1}{k+1}-\frac{1}{k+2}\right)$

$S_k = \frac{1}{2}-\frac{1}{k+2}$

f) $\sum_{n=1}^{\infty}\frac{1}{n+1}-\frac{1}{n+2} = \lim_{k\to\infty} S_k = \lim_{k\to\infty} S_k\left(\frac{1}{2}-\frac{1}{k+2}\right) = \frac{1}{2}-0 = \frac{1}{2}$

g) So Telescoping Series $\sum_{n=1}^{\infty}\frac{1}{n+1}-\frac{1}{n+2}$ converges to $\frac{1}{2}$

13) $\sum_{n=2}^{\infty}\frac{2}{3n+1}$. Explain why this series diverges. Use Integral Test.

Explanation:

$$\int_2^{\infty}\frac{2}{3x+1}dx = 2\int_2^{\infty}\frac{1}{3x+1}dx = 2\left[\frac{1}{3}\ln|3x+1|\right]\Bigg|_2^{\infty} = \frac{2}{3}\ln(\infty)-\frac{2}{3}\ln(7) = \infty$$

By Integral Test $\sum_{n=2}^{\infty}\frac{2}{3n+1}$ diverges because $\int_2^{\infty}\frac{2}{3x+1}dx = \infty$.

14) $\sum_{n=1}^{\infty} \frac{1}{\sqrt[7]{n^5}}$. Explain why this series diverges.

Explanation:

$\sum_{n=1}^{\infty} \frac{1}{\sqrt[7]{n^5}} = \sum_{n=1}^{\infty} \frac{1}{n^{5/7}}$ is a p-series with p = 5/7 < 1.

By p-series Theorem $\sum_{n=1}^{\infty} \frac{1}{\sqrt[7]{n^5}}$ converges because p < 1.

15) $\sum_{n=1}^{\infty} \left(\frac{1}{n^{5/4}} \right)$. Explain why this series converges.

Explanation:

$\sum_{n=1}^{\infty} \left(\frac{1}{n^{5/4}} \right)$ is a p-series with p =5/4 >1;

Therefore, by p-series Test $\sum_{n=1}^{\infty} \left(\frac{1}{n^{5/4}} \right)$ converges.

16) $\sum_{n=1}^{\infty} \left(\frac{1}{3^n} \right)$. Explain why this series converges.

Explanation:

$\sum_{n=1}^{\infty} \left(\frac{1}{3^n} \right) = \sum_{n=1}^{\infty} \left(\frac{1}{3} \right)^n$ is a Geometric series with r = 1/3 and -1 < r < 1.

Therefore, by Geometric Series Theorem, $\sum_{n=1}^{\infty} \left(\frac{1}{3^n} \right)$ converges.

17) $\sum_{n=1}^{\infty}\left(\frac{1}{n^4}-\frac{1}{n}\right)$.   Explain why this series diverges.

Explanation:

$$\sum_{n=1}^{\infty}\left(\frac{1}{n^4}-\frac{1}{n}\right)=\sum_{n=1}^{\infty}\left(\frac{1}{n^4}\right)-\sum_{n=1}^{\infty}\left(\frac{1}{n}\right).$$

$\sum_{n=1}^{\infty}\left(\frac{1}{n^4}\right)$ is a p-series with p = 4; hence $\sum_{n=1}^{\infty}\left(\frac{1}{n^4}\right)$ converges to a finite number.

$\sum_{n=1}^{\infty}\left(\frac{1}{n}\right)$ is a p-series with p = 1; hence $\sum_{n=1}^{\infty}\left(\frac{1}{n}\right)$ diverges to ∞.

$$\sum_{n=1}^{\infty}\left(\frac{1}{n^2}-\frac{1}{n}\right)=\sum_{n=1}^{\infty}\left(\frac{1}{n^2}\right)-\sum_{n=1}^{\infty}\left(\frac{1}{n}\right)=(\text{finite value}) - \infty = -\infty$$

Therefore, $\sum_{n=1}^{\infty}\left(\frac{1}{n^4}-\frac{1}{n}\right)$ diverges to -∞.

18) $\sum_{n=1}^{\infty}\left(\frac{\ln n}{n^3}\right)$.   Use Basic Comparison Test to explain why this series converges.

Let $a_n=\frac{\ln n}{n^3}$ and $b_n=\frac{n}{n^3}=\frac{1}{n^2}$

Hence, $a_n=\frac{\ln n}{n^3}<b_n=\frac{n}{n^3}=\frac{1}{n^2}$

$\sum_{n=1}^{\infty}(b_n)=\sum_{n=1}^{\infty}\frac{1}{n^2}$ is a p-series with p=2; hence it converges to a finite number.

$$\sum_{n=1}^{\infty}\left(\frac{\ln n}{n^3}\right)<\sum_{n=1}^{\infty}\left(\frac{1}{n^2}\right)=\text{finite number.}$$

Therefore, $\sum_{n=1}^{\infty}\left(\frac{\ln n}{n^3}\right)$ converges.

19) $\sum_{n=2}^{\infty}\left(\frac{1}{\sqrt[3]{n}-1}\right)$. Explain why this series diverges. Use Limit Comparison Test.

Explanation:

Let $a_n = \frac{1}{\sqrt[3]{n}-1} = \frac{1}{n^{1/3}-1}$; $b_n = \frac{1}{\sqrt[3]{n}} = \frac{1}{n^{1/3}}$

Note: $\sum_{n=2}^{\infty}(b_n) = \sum_{n=2}^{\infty}\left(\frac{1}{n^{1/3}}\right)$ is a p-series with p=1/3; hence it diverges.

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty}\left(\frac{a_n}{b_n}\right) = \lim_{n\to\infty}\left(\frac{\frac{1}{n^{1/3}-1}}{\frac{1}{n^{1/3}}}\right) = \lim_{n\to\infty}\left(\frac{n^{1/3}}{n^{1/3}-1}\right) = \lim_{n\to\infty}\left(\frac{\frac{1}{3}n^{-2/3}}{\frac{1}{3}n^{-2/3}}\right) = 1$$

Because $\lim_{n\to\infty}\left(\frac{a_n}{b_n}\right) = 1 > 0$ Limit Comparison Test says that either

both $\sum_{n=2}^{\infty}(a_n)$ and $\sum_{n=2}^{\infty}(b_n)$ converges or both diverges.

Since $\sum_{n=2}^{\infty}\left(\frac{1}{n^{1/3}}\right)$ diverges, $\sum_{n=2}^{\infty}\left(\frac{1}{\sqrt[3]{n}-1}\right)$ also diverges.

20) $\sum_{n=1}^{\infty}\left(\frac{n}{\sqrt{n^4+3n}}\right)$. Explain why this series diverges. Use Limit Comparison Test.

Explanation:

Let $a_n = \frac{n}{\sqrt{n^4+3n}}$; $b_n = \frac{n}{\sqrt{n^4}} = \frac{n}{n^2} = \frac{1}{n}$

Note: $\sum_{n=1}^{\infty}(b_n) = \sum_{n=1}^{\infty}\left(\frac{1}{n}\right)$ is a p-series with p = 1; hence it diverges.

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty}\left(\frac{a_n}{b_n}\right) = \lim_{n\to\infty}\left(\frac{\frac{n}{\sqrt{n^4+3n}}}{\frac{1}{n}}\right) = \lim_{n\to\infty}\left(\frac{n^2}{\sqrt{n^4+3n}}\right) = 1$$

Because $\lim_{n\to\infty}\left(\frac{a_n}{b_n}\right) = 1 > 0$ Limit Comparison Test says that either

both $\sum_{n=2}^{\infty}(a_n)$ and $\sum_{n=2}^{\infty}(b_n)$ converges or both diverges.

Since $\sum_{n=1}^{\infty}\left(\frac{1}{n}\right)$ diverges, $\sum_{n=1}^{\infty}\left(\frac{n}{\sqrt{n^4+3n}}\right)$ also diverges.

21) $\sum_{n=1}^{\infty}\left(\frac{1}{\sqrt{n^3+5n}}\right)$. Explain why this series converges. Use Limit Comparison Test.

Explanation:

Let $a_n = \frac{1}{\sqrt{n^3+5n}}$; $b_n = \frac{1}{\sqrt{n^3}} = \frac{1}{n^{3/2}}$

Note: $\sum_{n=1}^{\infty}(b_n) = \sum_{n=1}^{\infty}\left(\frac{1}{n^{3/2}}\right)$ is a p-series with p = 3/2; hence it converges.

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty}\left(\frac{a_n}{b_n}\right) = \lim_{n\to\infty}\left(\frac{\frac{1}{\sqrt{n^3+5n}}}{\frac{1}{n^{3/2}}}\right) = \lim_{n\to\infty}\left(\frac{n^{3/2}}{\sqrt{n^3+5n}}\right) = 1$$

Because $\lim_{n\to\infty}\left(\frac{a_n}{b_n}\right) = 1 > 0$ Limit Comparison Test says that either

both $\sum_{n=2}^{\infty}(a_n)$ and $\sum_{n=2}^{\infty}(b_n)$ converges or both diverges.

Since $\sum_{n=1}^{\infty}(b_n) = \sum_{n=1}^{\infty}\left(\frac{1}{n^{3/2}}\right)$ converges, $\sum_{n=1}^{\infty}\left(\frac{1}{\sqrt{n^3+5n}}\right)$ also converges.

22) $\sum_{n=1}^{\infty}\left(\frac{n+4}{n(n+5)}\right)$. Explain why this series diverges. Use Limit Comparison Test.

Explanation:

Let $a_n = \frac{n+4}{n(n+5)}$; $b_n = \frac{n}{n(n)} = \frac{1}{n}$

Note: $\sum_{n=1}^{\infty}(b_n) = \sum_{n=1}^{\infty}\left(\frac{1}{n}\right)$ is a p-series with p = 1; hence it diverges.

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty}\left(\frac{a_n}{b_n}\right) = \lim_{n\to\infty}\left(\frac{\frac{n+4}{n(n+5)}}{\frac{1}{n}}\right) = \lim_{n\to\infty}\left(\frac{n^2+4}{n^2+5n}\right) = 1$$

Because $\lim_{n\to\infty}\left(\frac{a_n}{b_n}\right) = 1 > 0$ Limit Comparison Test says that either

both $\sum_{n=2}^{\infty}(a_n)$ and $\sum_{n=2}^{\infty}(b_n)$ converges or both diverges.

Since $\sum_{n=1}^{\infty}(b_n) = \sum_{n=1}^{\infty}\left(\frac{1}{n}\right)$ diverges, $\sum_{n=1}^{\infty}\left(\frac{n+4}{n(n+5)}\right)$ also diverges.

23) $\sum_{n=1}^{\infty}\left(\frac{1}{3^n-7}\right)$. Explain why this series converges. Use Limit Comparison Test.

Explanation:

Let $a_n = \frac{1}{3^n-7}$; $b_n = \frac{1}{3^n}$

Note: $\sum_{n=1}^{\infty}(b_n) = \sum_{n=1}^{\infty}\left(\frac{1}{3^n}\right) = \sum_{n=1}^{\infty}\left(\frac{1}{3}\right)^n$ is a Geometric series with r = 1/3; hence it converges.

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty}\left(\frac{a_n}{b_n}\right) = \lim_{n\to\infty}\left(\frac{\frac{1}{3^n-7}}{\frac{1}{3^n}}\right) = \lim_{n\to\infty}\left(\frac{3^n}{3^n-7}\right) = \lim_{n\to\infty}\left(\frac{(\ln 3)\cdot 3^n}{(\ln 3)\cdot 3^n}\right) = 1$$

Because $\lim_{n\to\infty}\left(\frac{a_n}{b_n}\right) = 1 > 0$ Limit Comparison Test says that either

both $\sum_{n=2}^{\infty}(a_n)$ and $\sum_{n=2}^{\infty}(b_n)$ converges or both diverges.

Since $\sum_{n=1}^{\infty}(b_n) = \sum_{n=1}^{\infty}\left(\frac{1}{3^n}\right) = \sum_{n=1}^{\infty}\left(\frac{1}{3}\right)^n$ converges, $\sum_{n=1}^{\infty}\left(\frac{1}{3^n-7}\right)$ also converges.

24) $\sum_{n=1}^{\infty}\left(\frac{(-1)^n}{n^4}\right)$. Explain why this series converges. Use Alternating Series Test.

Explanation:

Let $a_n = \frac{1}{n^4}$.

a) Show $\lim_{n\to\infty} a_n = 0$: $\lim_{n\to\infty} a_n = \lim_{n\to\infty}\left(\frac{1}{n^4}\right) = 0$

b) Show a_n is decreasing or show $a_{n+1} \le a_n$

$a_n = \frac{1}{n^4}$ and $a_{n+1} = \frac{1}{(n+1)^4}$;

Note: $\frac{1}{(n+1)^4} < \frac{1}{n^4}$; so $a_{n+1} \le a_n$

Hence, $a_n = \frac{1}{n^4}$ is decreasing

Therefore, by Alternating Series Test, $\sum_{n=1}^{\infty}\left(\frac{(-1)^n}{n^4}\right)$ converges.

25) $\sum_{n=1}^{\infty}\left(\frac{(-1)^n (n+3)}{n^2+4}\right)$. Explain why this series converges. Use Alternating Series Test.

Explanation:

Let $a_n = \frac{n+3}{n^2+4}$.

a) Show $\lim_{n\to\infty} a_n = 0$: $\lim_{n\to\infty} a_n = \lim_{n\to\infty}\left(\frac{n+3}{n^2+4}\right) = \lim_{n\to\infty}\left(\frac{1}{2n}\right) = 0$

b) Show a_n is decreasing.

a_1=4/5; a_2=5/8; $a_3 = 6/13$; $a_4 = 7/20$; $a_5 = 9/40$

Also, for $f(x) = \frac{x+3}{x^2+4}$, $x \geq 1$;

$$f'(x) = \frac{(x^2+4)(1)-(x+3)(2x)}{\left(x^2+4\right)^2} = \frac{x^2+4-2x^2-6x}{\left(x^2+1\right)^2} = \frac{-x^2-6x+4}{\left(x^2+1\right)^2} < 0 \text{ for } x \geq 1.$$

Hence, $a_n = \frac{n+3}{n^2+4}$ is decreasing

Therefore, by Alternating Series Test, $\sum_{n=1}^{\infty}\left(\frac{(-1)^n (n+3)}{n^2+4}\right)$ converges.

26) $\sum_{n=2}^{\infty}\left(\frac{(-1)^n(n)}{n^2-5}\right)$. Explain why this series converges. Use Alternating Series Test.

Explanation:

Let $a_n = \frac{n}{n^3-5}$.

a) Show $\lim_{n\to\infty} a_n = 0$: $\lim_{n\to\infty} a_n = \lim_{n\to\infty}\left(\frac{n}{n^3-5}\right) = \lim_{n\to\infty}\left(\frac{1}{3n^2}\right) = 0$

b) Show a_n is decreasing.

Also, for $f(x) = \frac{x}{x^3-5}$, x ≥ 2; $f'(x) = \frac{(x^3-5)(1)-(x)(3x^2)}{\left(x^3-5\right)^2} = \frac{x^3-5-3x^3}{\left(x^3-5\right)^2} = \frac{-2x^3-5}{\left(x^2+1\right)^2} < 0$ for x ≥ 2.

Hence, $a_n = \frac{n}{n^3-3}$ is decreasing

Therefore, by Alternating Series Test, $\sum_{n=2}^{\infty}\left(\frac{(-1)^n(n)}{n^2-5}\right)$ converges.

27) $\sum_{n=1}^{\infty}\left(\frac{(-1)^n\sqrt{n}}{n+4}\right)$. Explain why this series converges. Use Alternating Series Test.

Explanation:

Let $a_n = \frac{\sqrt{n}}{n+4}$.

a) Show $\lim_{n\to\infty} a_n = 0$: $\lim_{n\to\infty} a_n = \lim_{n\to\infty}\left(\frac{\sqrt{n}}{n+4}\right) = \lim_{n\to\infty}\left(\frac{(1/2)n^{-1/2}}{1}\right) = \lim_{n\to\infty}\left(\frac{1}{2n^{1/2}}\right) = 0$

b) Show a_n is decreasing.

$a_1 = 1/2$; $a_2 = \frac{\sqrt{2}}{3}$; $a_3 = \frac{\sqrt{3}}{4}$; $a_4 = \frac{\sqrt{4}}{5}$;

Also, for $f(x) = \frac{\sqrt{x}}{x+4}$ $x \geq 1$;

$$f'(x) = \frac{(x+4)\left((1/2)x^{-1/2}\right) - \sqrt{x}}{(x+4)^2} = \frac{\frac{x+1}{2\sqrt{x}} - \sqrt{x}}{(x+4)^2} = \frac{\frac{(x+4)-2x}{2\sqrt{x}}}{(x+4)^2} = \frac{\frac{-x+4}{2\sqrt{x}}}{(x+4)^2} < 0 \text{ for } x \geq 2.$$

Hence, $a_n = \frac{\sqrt{n}}{n+4}$ is decreasing

Therefore, by Alternating Series Test, $\sum_{n=1}^{\infty}\left(\frac{(-1)^n\sqrt{n}}{n+4}\right)$ converges.

28) $\sum_{n=4}^{\infty}\left(\frac{(-1)^n n}{n-5}\right)$. Use *n*th Term Test to explain why this series diverges.

Explanation:

Let $a_n = \frac{n}{n-5}$.

$a)$ Show $\lim_{n\to\infty} a_n = 0$: $\lim_{n\to\infty} a_n = \lim_{n\to\infty}\left(\frac{n}{n-5}\right) = \lim_{n\to\infty}\left(\frac{1}{1}\right) = 1$

Therefore, $\sum_{n=4}^{\infty}\left(\frac{(-1)^n n}{n-5}\right)$ diverges by nth Term Test.

29) $\sum_{n=2}^{\infty}\left(\frac{(-1)^n \ln n^3}{n}\right)$. Explain why this series converges.

Explanation:

Let $a_n = \frac{\ln n^3}{n}$.

$a)$ Show $\lim_{n\to\infty} a_n = 0$: $\lim_{n\to\infty} a_n = \lim_{n\to\infty}\left(\frac{\ln n^3}{n}\right) = \lim_{n\to\infty}\left(\frac{3\ln n}{n}\right) = \lim_{n\to\infty}\left(\frac{3(1/n)}{1}\right) = \lim_{n\to\infty}\left(\frac{3}{n}\right) = 0$

$b)$ Let $f(x) = \frac{\ln x^3}{x} = \frac{3\ln x}{x}$; $f'(x) = \frac{(x)(3/x) - (3\ln x)(1)}{x^2} = \frac{3-3\ln x}{x^2} < 0$ for $x \geq 3$

Hence, $a_n = \frac{\ln n^3}{n}$ is decreasing.

Therefore, by Alternating Series Test, $\sum_{n=2}^{\infty}\left(\frac{(-1)^n \ln n^3}{n}\right)$ converges.

30) $\sum_{n=1}^{\infty}\left(\frac{5n-1}{2n+5}\right)^{n}$. Explain why this series diverges by using Root Test.

Explanation:

Let $a_n = \left(\frac{5n-1}{2n+5}\right)^{n}$. $\sqrt[n]{a_n} = \sqrt[n]{\left(\frac{5n-1}{2n+5}\right)^{n}} = \frac{5n-1}{2n+5}$

$$L = \lim_{n\to\infty}\sqrt[n]{a_n} = \lim_{n\to\infty}\left(\frac{5n-1}{2n+5}\right) = \lim_{n\to\infty}\left(\frac{5}{2}\right) = \frac{5}{2} > 1$$

Therefore, $\sum_{n=1}^{\infty}\left(\frac{5n-1}{2n+5}\right)$ diverges by Root Test because $L > 1$.

31) $\sum_{n=1}^{\infty}\left(\frac{4n}{9n-1}\right)^{n}$. Explain why this series converges by Root Test.

Explanation:

Let $a_n = \left(\frac{4n}{9n-1}\right)^{n}$. $\sqrt[n]{a_n} = \sqrt[n]{\left(\frac{4n}{9n-1}\right)^{n}} = \frac{4n}{9n-1}$

$$L = \lim_{n\to\infty} a_n = \lim_{n\to\infty}\left(\frac{4n}{9n-1}\right) = \lim_{n\to\infty}\left(\frac{4}{9}\right) = \frac{4}{9}$$

Therefore, $\sum_{n=1}^{\infty}\left(\frac{4n}{9n-1}\right)^{n}$ converges by Root Test because $L < 1$.

32) $\sum_{n=1}^{\infty}\left(\frac{4n}{e^{n^2}}\right)$.   Explain why this series converges by Ratio Test.

Explanation:

Let $a_n = \frac{4n}{e^{n^2}}$; $a_{n+1} = \frac{4n+4}{e^{(n+1)^2}} = \frac{4n+4}{e^{n^2+2n+1}} = \frac{4n+4}{e^{n^2}e^{2n+1}}$

$$L = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{4n+4}{e^{n^2}e^{2n+1}}\cdot\frac{e^{n^2}}{4n}\right| = \lim_{n\to\infty}\left(\frac{4n+1}{4n}\cdot\frac{1}{e^{2n+1}}\right) = \lim_{n\to\infty}\left(\frac{4n+1}{4n}\right)\cdot\lim_{n\to\infty}\left(\frac{1}{e^{2n+1}}\right) = (1)(0) = 0$$

Therefore, $\sum_{n=1}^{\infty}\left(\frac{n}{e^{n^2}}\right)$ converges by Root Test because $L<1$.

33) $\sum_{n=1}^{\infty}\left(\frac{5^n}{n^3}\right)$.   Explain why this series diverges by Root Test.

Explanation:

Let $a_n = \frac{5^n}{n^3}$; $a_{n+1} = \frac{5^{n+1}}{(n+1)^3} = \frac{5^n 5^1}{(n+1)^3}$

$$L = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{5^n5^1}{(n+1)^3}\cdot\frac{n^3}{5^n}\right| = \lim_{n\to\infty}\left(\frac{5^1 n^3}{(n+1)^3}\right) = \lim_{n\to\infty}\frac{5n^3}{(n+1)^3} = 5$$

Therefore, $\sum_{n=1}^{\infty}\left(\frac{5^n}{n^3}\right)$ diverges by Ratio Test because $L>1$.

34) $\sum_{n=1}^{\infty}\left(\frac{n!}{e^n}\right)$. Explain why this series diverges by Ratio Test.

Explanation:

Let $a_n = \frac{n!}{e^n}$; $a_{n+1} = \frac{(n+1)!}{e^{n+1}} = \frac{(n+1)\left[(n)(n-1)\cdots(3)(2)(1)\right]}{e^n e^1} = \frac{(n+1)[n!]}{e^n e^1}$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)[n!]}{e^n e^1}\cdot\frac{e^n}{n!}\right| = \lim_{n\to\infty}\left((n+1)\cdot\frac{1}{e^1}\right) = \lim_{n\to\infty}(n+1)\cdot\lim_{n\to\infty}\left(\frac{1}{e}\right) = (\infty)\left(\frac{1}{e}\right) = \infty$$

Therefore, $\sum_{n=1}^{\infty}\left(\frac{n!}{e^n}\right)$ diverges by Ratio Test.

35) $f(x) = e^{-2x}$. Write a 3rd degree polynomial to approximate $f(x)$ values around $c = 0$.

$f(x) = e^{-2x}$ $\qquad f(0) = e^0 = 1$

$f'(x) = -2e^{-2x}$ $\qquad f'(0) = -2e^0 = -2$

$f''(x) = 4e^{-2x}$ $\qquad f''(0) = 4e^0 = 4$

$f'''(x) = -8e^{-2x}$ $\qquad f'''(0) = -8e^0 = -8$

$$P_3(x) = f(c) + f'(c)(x-c) + \frac{f''(c)(x-c)^2}{2!} + \frac{f'''(c)(x-c)^3}{3!}$$

$$P_3(x) = 1 + (-2)(x) + \frac{(4)(x)^2}{2} + \frac{(-8)(x)^3}{6}$$

36) $f(x)=\cos\pi x.$ Write a 4th degree polynomial to approximate $f(x)$ values around $c=0.$

$f(x)=\cos\pi x \qquad f(0)=1$

$f'(x)=-\pi\sin\pi x \qquad f'(0)=0$

$f''(x)=-\pi^2\cos\pi x \qquad f''(0)=-\pi^2$

$f'''(x)=\pi^3\sin\pi x \qquad f'''(0)=0$

$f^{(4)}(x)=\pi^4\cos\pi x \qquad f'''(0)=\pi^4$

$$P_4(x)=f(c)+f'(c)(x-c)+\frac{f''(c)(x-c)^2}{2!}+\frac{f'''(c)(x-c)^3}{3!}+\frac{f^{(4)}(c)(x-c)^4}{4!}$$

$$P_4(x)=1+(0)(x)+(0)(x)+\frac{(-\pi^2)(x)^2}{2}+\frac{(0)(x)^3}{6}+\frac{(\pi^4)(x)^4}{24}$$

37) $f(x)=e^{-3x}.$ Write a 3rd degree polynomial to approximate $f(x)$ values around $c=0.$

$f(x)=e^{-3x} \qquad f(0)=e^0=1$

$f'(x)=-3e^{-3x} \qquad f'(0)=-3e^0$ = -3

$f''(x)=9e^{-3x} \qquad f''(0)=9e^0$ = 9

$f'''(x)=-27e^{-3x} \qquad f'''(0)=-27e^0$ = -27

$$P_3(x)=f(c)+f'(c)(x-c)+\frac{f''(c)(x-c)^2}{2!}+\frac{f'''(c)(x-c)^3}{3!}$$

$$P_3(x)=1+(-3)(x)+\frac{(9)(x)^2}{2}+\frac{(-27)(x)^3}{6}$$

38) $f(x)=\tan x$. Write a 3rd degree polynomial to approximate $f(x)$ values around $c=-\dfrac{\pi}{4}$.

$$f(x)=\tan x \qquad f\left(-\frac{\pi}{4}\right)=-1$$

$$f'(x)=\sec^2 x \qquad f'\left(-\frac{\pi}{4}\right)=2$$

$$f''(x)=2\sec^2 x\tan x \qquad f''\left(-\frac{\pi}{4}\right)=-4$$

$$f'''(x)=4\sec^2 x\tan^2 x+2\sec^4 x \qquad f'''\left(-\frac{\pi}{4}\right)=16$$

$$P_3(x)=f(c)+f'(c)(x-c)+\frac{f''(c)(x-c)^2}{2!}+\frac{f'''(c)(x-c)^3}{3!}$$

$$P_3(x)=1+(2)(x--\frac{\pi}{4})+\frac{(-4)(x--\frac{\pi}{4})^2}{2}+\frac{(16)(x--\frac{\pi}{4})^3}{6}$$

$$P_3(x)=1+(2)\left(x+\frac{\pi}{4}\right)+\frac{(-4)\left(x+\frac{\pi}{4}\right)^2}{2}+\frac{(16)\left(x+\frac{\pi}{4}\right)^3}{6}$$

39) $\sum_{n=0}^{\infty}\left(\frac{x}{10}\right)^{n}$. Find interval of convergence for this series.

Let $u_n = \left(\frac{x}{10}\right)^{n}$ and $u_{n+1} = \left(\frac{x}{10}\right)^{n+1} = \left(\frac{x}{10}\right)^{n}\left(\frac{x}{10}\right)^{1}$

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty} a_n \left|\frac{\left(\frac{x}{10}\right)^{n}\left(\frac{x}{10}\right)^{1}}{\left(\frac{x}{10}\right)^{n}}\right| = \left|\left(\frac{x}{10}\right)\right|$$

Set $\left|\left(\frac{x}{10}\right)\right| < 1.$

Hence, $-1 < \frac{x}{10} < 1 \quad \Leftrightarrow \quad -10 < x < 10$

Therefore, interval of convergence is (-10, 10).

40) $\sum_{n=0}^{\infty}\frac{(-1)^n(x-2)^n}{(n+1)^2}$. Find interval of convergence for this series.

Let $u_n=\frac{(-1)^n(x-2)^n}{(n+1)^2}$ and $u_{n+1}=\frac{(-1)^{n+1}(x-2)^{n+1}}{((n+1)+1)^2}=\frac{(-1)^{n+1}(x-2)^n(x-2)^1}{(n+2)^2}$

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right|=\lim_{n\to\infty}\left|\frac{(-1)^{n+1}(x-2)^n(x-2)^1}{(n+2)^2}\cdot\frac{(n+1)^2}{(-1)^n(x-2)^n}\right|=\lim_{n\to\infty}\left|\frac{(n+1)^2}{(n+2)^2}\cdot(x-2)^1\right|$$

$$=\lim_{n\to\infty}\left|\frac{(n+1)^2}{(n+2)^2}\right|\cdot\lim_{n\to\infty}|x-2|=1\cdot|x-2|=|x-2|$$

Set $|x-2|<1$.

Hence, $-1<x-2<1 \iff 1<x<3$

Note when $x=1$, $\sum_{n=0}^{\infty}\frac{(-1)^n(x-2)^n}{(n+1)^2}=\sum_{n=0}^{\infty}\frac{(-1)^n(-1)^n}{(n+1)^2}=\sum_{n=0}^{\infty}\frac{1}{(n+1)^2}$ converges by Integral Test.

Note when $x=3$, $\sum_{n=0}^{\infty}\frac{(-1)^n(x-2)^n}{(n+1)^2}=\sum_{n=0}^{\infty}\frac{(-1)^n(1)^n}{(n+1)^2}=\sum_{n=0}^{\infty}\frac{(-1)^n}{(n+1)^2}$ converges by Alternating Series Test.

Therefore, interval of convergence is [1, 3].

41) $\sum_{n=0}^{\infty}n!(x-2)^n$. Find interval of convergence for this series.

Let $u_n=n!(x-2)^n$ and $u_{n+1}=(n+1)!(x-2)^{n+1}=(n+1)(n)!(x-2)^n(x-2)^1$

Note that $(n+1)!=(n+1)\cdot\left[(n)(n-1)(n-2)\cdots(3)(2)(1)\right]=(n+1)(n)!$

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right|=\lim_{n\to\infty}\left|\frac{(n+1)(n)!(x-2)^n(x-2)^1}{n!(x-2)^n}\right|=\lim_{n\to\infty}|(n+1)(x-2)|$$

$$=\lim_{n\to\infty}|(n+1)|\cdot\lim_{n\to\infty}|(x-2)|=\infty\cdot\lim_{n\to\infty}|(x-2)|=\infty$$

Therefore, $\sum_{n=0}^{\infty}n!(x-2)^n$ only converges when $x=2$.

When $x=2$, $\sum_{n=0}^{\infty}n!(x-2)^n=\sum_{n=0}^{\infty}n!(0)^n=\sum_{n=0}^{\infty}0=0$.

42) $f(x) = \dfrac{2}{3-x}$. Find a power series representation for $f(x)$ centered at c = 0.

Note: $\dfrac{2}{3-x} = \dfrac{2/3}{3/3 - x/3} = \dfrac{2/3}{1 - x/3} = \dfrac{a}{1-\ r}$

So power series representation for $f(x)$ is $\sum_{\text{n=0}}^{\infty} ar^n = \sum_{\text{n=0}}^{\infty} \left(\dfrac{2}{3}\right)\left(\dfrac{x}{3}\right)^n$

43) $f(x) = \dfrac{3}{2+x}$. Find a power series representation for $f(x)$ centered at c = 0.

Note: $\dfrac{3}{2+x} = \dfrac{3/2}{2/2 - (-x)} = \dfrac{3/2}{1-(-x)} = \dfrac{a}{1-\ r}$

So power series representation for $f(x)$ is $\sum_{\text{n=0}}^{\infty} ar^n = \sum_{\text{n=0}}^{\infty} \left(\dfrac{3}{2}\right)(-x)^n$

44) $f(x) = \dfrac{6}{4-x}$. Find a power series representation for $f(x)$ centered at c = 1.

Note: $\dfrac{6}{4-x} = \dfrac{6}{4-(x-1)-1} = \dfrac{6}{3-(x-1)} = \dfrac{6/3}{3/3-(x-1)/3} = \dfrac{2}{1-(x-1)/3} = \dfrac{a}{1-\ r}$

So power series representation for $f(x)$ is $\sum_{\text{n=0}}^{\infty} ar^n = \sum_{\text{n=0}}^{\infty} 2\left(\dfrac{x-1}{3}\right)^n$

45) $f(x) = \dfrac{1}{3-2x}$. Find a power series representation for $f(x)$ centered at $c = 2$.

$Note$: $3 - 2x = 3 - 2(x-2) - 4$

Note: $\dfrac{1}{3-2x} = \dfrac{1}{3-2(x-2)-4} = \dfrac{1/-1}{-1-2(x-2)} = \dfrac{-1}{\dfrac{-1}{-1} - \dfrac{2(x-2)}{-1}} = \dfrac{-1}{1 - -2(x-2)} = \dfrac{a}{1-\ r}$

So power series representation for $f(x)$ is $\sum_{\text{n=0}}^{\infty} ar^n = \sum_{\text{n=0}}^{\infty} (-1)\left[-2(x-2)\right]^n$