Test 2 Review

9) $a_n = \dfrac{5n+2}{n}$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty}\left(\frac{5n+2}{n}\right) = \lim_{n\to\infty}\left(\frac{5}{1}\right) = 5$$

10) $a_n = \sin\dfrac{n\pi}{2}$

For $n = 1$, $a_1 = \sin\dfrac{1\pi}{2} = 1$

For $n = 2$, $a_2 = \sin\dfrac{2\pi}{2} = 0$

For $n = 3$, $a_3 = \sin\dfrac{3\pi}{2} = -1$

For $n = 4$, $a_4 = \sin\dfrac{4\pi}{2} = 0$

Hence sequence of $\{a_n\} = \{1, 0, -1, 0, 1, 0, -1, 0, 1, 0, -1, 0, 1, 0, -1, 0, 1, 0, -1, 0, 1, 0, -1, 0, ...\}$

Therefore $\lim\limits_{n\to\infty} a_n$ diverges.

11) $a_n = \left(\dfrac{2}{5}\right)^n + 5$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty}\left(\left(\frac{2}{5}\right)^n + 5\right) = \lim_{n\to\infty}\left(\frac{2}{5}\right)^n + \lim_{n\to\infty}(5) = 0 + 5 = 5$$

15) $a_n = \dfrac{n}{n^2+1}$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty}\left(\frac{n}{n^2+1}\right) = \lim_{n\to\infty}\left(\frac{1}{2n}\right) = \frac{1}{\infty} = 0$$

17) $a_n = \sqrt{n+1} - \sqrt{n}$

Note: $\sqrt{n+1} - \sqrt{n} = \left(\sqrt{n+1} - \sqrt{n}\right)\left(\frac{\sqrt{n+1} + \sqrt{n}}{\sqrt{n+1} + \sqrt{n}}\right) = \frac{n+1-n}{\sqrt{n+1} + \sqrt{n}} = \frac{1}{\sqrt{n+1} + \sqrt{n}}$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty}\left(\sqrt{n+1} - \sqrt{n}\right) = \lim_{n\to\infty}\left(\frac{1}{\sqrt{n+1} + \sqrt{n}}\right) = \frac{1}{\infty} = 0$$

27) $\sum_{n=1}^{\infty}\left(\frac{3}{2}\right)^{n-1}$

For $n = 5$, $S_5 = \sum_{n=1}^{5}\left(\frac{3}{2}\right)^{n-1} = \left(\frac{3}{2}\right)^0 + \left(\frac{3}{2}\right)^1 + \left(\frac{3}{2}\right)^2 + \left(\frac{3}{2}\right)^3 + \left(\frac{3}{2}\right)^4$

For $n = 10$, $S_{10} = \sum_{n=1}^{10}\left(\frac{3}{2}\right)^{n-1} = \left(\frac{3}{2}\right)^0 + \left(\frac{3}{2}\right)^1 + \left(\frac{3}{2}\right)^2 + \cdots + \left(\frac{3}{2}\right)^9 + \left(\frac{3}{2}\right)^{10}$

For $n = 15$, $S_{15} = \sum_{n=1}^{15}\left(\frac{3}{2}\right)^{n-1} = \left(\frac{3}{2}\right)^0 + \left(\frac{3}{2}\right)^1 + \left(\frac{3}{2}\right)^2 + \cdots + \left(\frac{3}{2}\right)^{14} + \left(\frac{3}{2}\right)^{15}$

For $n = 20$, $S_{20} = \sum_{n=1}^{20}\left(\frac{3}{2}\right)^{n-1} = \left(\frac{3}{2}\right)^0 + \left(\frac{3}{2}\right)^1 + \left(\frac{3}{2}\right)^2 + \cdots + \left(\frac{3}{2}\right)^{19} + \left(\frac{3}{2}\right)^{20}$

For $n = 25$, $S_{25} = \sum_{n=1}^{25}\left(\frac{3}{2}\right)^{n-1} = \left(\frac{3}{2}\right)^0 + \left(\frac{3}{2}\right)^1 + \left(\frac{3}{2}\right)^2 + \cdots + \left(\frac{3}{2}\right)^{24} + \left(\frac{3}{2}\right)^{25}$

28) $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2n}$

For $n = 5$, $S_5 = \sum_{n=1}^{5} \frac{(-1)^{n+1}}{2n} = \frac{(-1)^2}{2(1)} + \frac{(-1)^3}{2(2)} + \frac{(-1)^4}{2(3)} + \frac{(-1)^5}{2(4)} + \frac{(-1)^6}{2(5)}$

For $n = 10$, $S_{10} = \sum_{n=1}^{10} \frac{(-1)^{n+1}}{2n} = \frac{(-1)^2}{2(1)} + \frac{(-1)^3}{2(2)} + \frac{(-1)^4}{2(3)} + \cdots + \frac{(-1)^{10}}{2(9)} + \frac{(-1)^{11}}{2(10)}$

For $n = 15$, $S_{15} = \sum_{n=1}^{15} \frac{(-1)^{n+1}}{2n} = \frac{(-1)^2}{2(1)} + \frac{(-1)^3}{2(2)} + \frac{(-1)^4}{2(3)} + \cdots + \frac{(-1)^{15}}{2(14)} + \frac{(-1)^{16}}{2(15)}$

For $n = 20$, $S_{20} = \sum_{n=1}^{20} \frac{(-1)^{n+1}}{2n} = \frac{(-1)^2}{2(1)} + \frac{(-1)^3}{2(2)} + \frac{(-1)^4}{2(3)} + \cdots + \frac{(-1)^{20}}{2(19)} + \frac{(-1)^{21}}{2(20)}$

For $n = 25$, $S_{25} = \sum_{n=1}^{25} \frac{(-1)^{n+1}}{2n} = \frac{(-1)^2}{2(1)} + \frac{(-1)^3}{2(2)} + \frac{(-1)^4}{2(3)} + \cdots + \frac{(-1)^{25}}{2(24)} + \frac{(-1)^{26}}{2(25)}$

31) $\sum_{n=0}^{\infty} \left(\frac{2}{5}\right)^n$. Explain why this series converges.

Explanation:

$\sum_{n=0}^{\infty} \left(\frac{2}{5}\right)^n$ is a Geometric Series with $r = \frac{2}{5}$.

By Geometric Series Theorem $\sum_{n=0}^{\infty} \left(\frac{2}{5}\right)^n$ converges because -1 < r < 1.

Also, $\sum_{n=0}^{\infty} \left(\frac{2}{5}\right)^n = \frac{a}{1-r} = \frac{1}{1-\frac{2}{5}} = \frac{5}{3}$

33) $\sum_{n=0}^{\infty}\left[(0.6)^n+(0.8)^n\right]$. Explain why this series converges.

Explanation:

$$\sum_{n=0}^{\infty}\left[(0.6)^n+(0.8)^n\right]=\sum_{n=0}^{\infty}\left[(0.6)^n\right] + \sum_{n=0}^{\infty}\left[+(0.8)^n\right]$$

$\sum_{n=0}^{\infty}\left[(0.6)^n\right]$ is a Geometric Series with $r=0.6$.

Hence $\sum_{n=0}^{\infty}\left[(0.6)^n\right]$ converges by Geometric Series Theorem.

$\sum_{n=0}^{\infty}\left[(0.8)^n\right]$ is a Geometric Series with $r=0.8$.

Hence $\sum_{n=0}^{\infty}\left[(0.8)^n\right]$ converges by Geometric Series Theorem

Therefore, $\sum_{n=0}^{\infty}\left[(0.6)^n+(0.8)^n\right]=\sum_{n=0}^{\infty}\left[(0.6)^n\right] + \sum_{n=0}^{\infty}\left[+(0.8)^n\right]$ converges.

37) $\sum_{n=0}^{\infty}(1.67)^n$. Explain why this series diverges.

Explanation:

$\sum_{n=0}^{\infty}(1.67)^n$ is a Geometric Series with $r=1.67$.

By Geometric Series Theorem $\sum_{n=0}^{\infty}(1.67)^n$ diverges because r > 1.

37) $\sum_{n=2}^{\infty}(1.67)^n$. Explain why this series diverges. Use nth Term Test.

Explanation:

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty}(1.67)^n = \infty$$

By nth Term Test $\sum_{n=2}^{\infty}(1.67)^n$ diverges because $\lim_{n\to\infty} a_n \neq 0$.

39) $\sum_{n=2}^{\infty}(-1)^n \dfrac{n}{\ln n}$. Explain why this series diverges. Use nth Term Test.

Explanation:

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty}\frac{n}{\ln n} = \lim_{n\to\infty}\left(\frac{1}{1/n}\right) = \lim_{n\to\infty}(n) = \infty$$

By nth Term Test $\sum_{n=2}^{\infty}(-1)^n \dfrac{n}{\ln n}$ diverges because $\lim_{n\to\infty} a_n \neq 0$.

40) $\sum_{n=0}^{\infty}\dfrac{2n+1}{3n+2}$. Explain why this series diverges. Use nth Term Test.

Explanation:

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty}\frac{2n+1}{3n+2} = \lim_{n\to\infty}\left(\frac{2}{3}\right) = \frac{2}{3}$$

By nth Term Test $\sum_{n=2}^{\infty}(-1)^n \dfrac{n}{\ln n}$ diverges because $\lim_{n\to\infty} a_n \neq 0$.

43) $\sum_{n=1}^{\infty} \frac{2}{6n+1}$. Explain why this series diverges. Use Integral Test.

Explanation:

$$\int_2^{\infty} \frac{2}{6x+1}dx = 2\int_2^{\infty} \frac{1}{6x+1}dx = 2\left[\frac{1}{6}\ln\left|6x+1\right|\right]_2^{\infty} = \frac{1}{3}\ln(\infty) - \frac{1}{3}\ln(7) = \infty$$

By Integral Test $\sum_{n=1}^{\infty} \frac{2}{6n+1}$ diverges because $\int_2^{\infty} \frac{2}{6x+1}dx = \infty$.

44) $\sum_{n=1}^{\infty} \frac{1}{\sqrt[4]{n^3}}$. Explain why this series diverges.

Explanation:

$\sum_{n=1}^{\infty} \frac{1}{\sqrt[4]{n^3}} = \sum_{n=1}^{\infty} \frac{1}{n^{3/4}}$ is a p-series with p = 3/4 < 1.

By p-series Theorem $\sum_{n=1}^{\infty} \frac{1}{\sqrt[4]{n^3}}$ because p < 1.

45) $\sum_{n=1}^{\infty} \left(\frac{1}{n^{5/2}}\right)$. Explain why this series converges.

Explanation:

$\sum_{n=1}^{\infty} \left(\frac{1}{n^{5/2}}\right)$ is a p-series with p =5/2 >1;

Therefore, by p-series Test $\sum_{n=1}^{\infty} \left(\frac{1}{n^{5/2}}\right)$ converges.

46) $\sum_{n=1}^{\infty}\left(\frac{1}{5^n}\right)$. Explain why this series converges.

Explanation:

$\sum_{n=1}^{\infty}\left(\frac{1}{5^n}\right)=\sum_{n=1}^{\infty}\left(\frac{1}{5}\right)^n$ is a Geometric series with r = 1/5 < 1.

Therefore, by Geometric Series Theorem, $\sum_{n=1}^{\infty}\left(\frac{1}{5^n}\right)$ converges.

47) $\sum_{n=1}^{\infty}\left(\frac{1}{n^2}-\frac{1}{n}\right)$. Explain why this series diverges.

Explanation:

$$\sum_{n=1}^{\infty}\left(\frac{1}{n^2}-\frac{1}{n}\right)=\sum_{n=1}^{\infty}\left(\frac{1}{n^2}\right)-\sum_{n=1}^{\infty}\left(\frac{1}{n}\right).$$

$\sum_{n=1}^{\infty}\left(\frac{1}{n^2}\right)$ is a p-series with p = 2; hence $\sum_{n=1}^{\infty}\left(\frac{1}{n^2}\right)$ converges to a finite number.

$\sum_{n=1}^{\infty}\left(\frac{1}{n}\right)$ is a p-series with p = 1; hence $\sum_{n=1}^{\infty}\left(\frac{1}{n}\right)$ diverges to ∞.

$$\sum_{n=1}^{\infty}\left(\frac{1}{n^2}-\frac{1}{n}\right)=\sum_{n=1}^{\infty}\left(\frac{1}{n^2}\right)-\sum_{n=1}^{\infty}\left(\frac{1}{n}\right)=(\text{finite value}) - \infty = -\infty$$

Therefore, $\sum_{n=1}^{\infty}\left(\frac{1}{n^2}-\frac{1}{n}\right)$ diverges to $-\infty$.

48) $\sum_{n=1}^{\infty}\left(\frac{\ln n}{n^4}\right)$. Explain why this series converges.

Explanation:

Let $a_n = \frac{\ln n}{n^4}$; let $b_n = \frac{n}{n^4} = \frac{1}{n^3}$

Hence, $a_n = \frac{\ln n}{n^4} < b_n = \frac{n}{n^4} = \frac{1}{n^3}$

$\sum_{n=1}^{\infty}(b_n) = \sum_{n=1}^{\infty}\frac{1}{n^3}$ is a p-series with p=3; hence it converges to a finite number.

$\sum_{n=1}^{\infty}\left(\frac{\ln n}{n^4}\right) < \sum_{n=1}^{\infty}\left(\frac{1}{n^3}\right) =$ finite number.

Therefore, $\sum_{n=1}^{\infty}\left(\frac{\ln n}{n^4}\right)$ converges.

49) $\sum_{n=2}^{\infty}\left(\frac{1}{\sqrt[3]{n}-1}\right)$. Explain why this series diverges. Use Limit Comparison Test.

Explanation:

Let $a_n = \frac{1}{\sqrt[3]{n}-1} = \frac{1}{n^{1/3}-1}$; $\quad b_n = \frac{1}{\sqrt[3]{n}} = \frac{1}{n^{1/3}}$

Note: $\sum_{n=2}^{\infty}(b_n) = \sum_{n=2}^{\infty}\left(\frac{1}{n^{1/3}}\right)$ is a p-series with p=1/3; hence it diverges.

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty}\left(\frac{a_n}{b_n}\right) = \lim_{n\to\infty}\left(\frac{\frac{1}{n^{1/3}-1}}{\frac{1}{n^{1/3}}}\right) = \lim_{n\to\infty}\left(\frac{n^{1/3}}{n^{1/3}-1}\right) = \lim_{n\to\infty}\left(\frac{\frac{1}{3}n^{-2/3}}{\frac{1}{3}n^{-2/3}}\right) = 1$$

Because $\lim_{n\to\infty}\left(\frac{a_n}{b_n}\right) = 1 > 0$ Limit Comparison Test says that either

both $\sum_{n=2}^{\infty}(a_n)$ and $\sum_{n=2}^{\infty}(b_n)$ converges or both diverges.

Since $\sum_{n=2}^{\infty}\left(\frac{1}{n^{1/3}}\right)$ diverges, $\sum_{n=2}^{\infty}\left(\frac{1}{\sqrt[3]{n}-1}\right)$ also diverges.

49) $\sum_{n=2}^{\infty}\left(\frac{1}{\sqrt[3]{n}-1}\right)$. Explain why this series diverges. Now use Direct Comparison Test.

Explanation:

Let $a_n = \frac{1}{\sqrt[3]{n}-1} = \frac{1}{n^{1/3}-1}$; $b_n = \frac{1}{\sqrt[3]{n}} = \frac{1}{n^{1/3}}$

Note that $\frac{1}{n^{1/3}} < \frac{1}{n^{1/3}-1}$; and $\sum \frac{1}{n^{1/3}} < \sum \frac{1}{n^{1/3}-1}$

$\sum_{n=2}^{\infty}\left(\frac{1}{n^{1/3}}\right)$ is a p-series with p=1/3; hence it diverges to ∞.

From $\sum \frac{1}{n^{1/3}} < \sum \frac{1}{n^{1/3}-1} \Rightarrow \infty < \sum \frac{1}{n^{1/3}-1}$.

Therefore, $\sum \frac{1}{n^{1/3}-1}$ diverges.

50) $\sum_{n=1}^{\infty}\left(\frac{n}{\sqrt{n^3+3n}}\right)$. Explain why this series diverges. Use Limit Comparison Test.

Explanation:

Let $a_n = \frac{n}{\sqrt{n^3+3n}}$; $b_n = \frac{n}{\sqrt{n^3}} = \frac{n}{n^{3/2}} = \frac{1}{n^{1/2}}$

Note: $\sum_{n=1}^{\infty}(b_n) = \sum_{n=1}^{\infty}\left(\frac{1}{n^{1/2}}\right)$ is a p-series with p=1/2; hence it diverges.

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty}\left(\frac{a_n}{b_n}\right) = \lim_{n\to\infty}\left(\frac{\frac{n}{\sqrt{n^3+3n}}}{\frac{1}{n^{1/2}}}\right) = \lim_{n\to\infty}\left(\frac{n^{3/2}}{\sqrt{n^3+3n}}\right) = 1$$

Because $\lim_{n\to\infty}\left(\frac{a_n}{b_n}\right) = 1 > 0$ Limit Comparison Test says that either

both $\sum_{n=2}^{\infty}(a_n)$ and $\sum_{n=2}^{\infty}(b_n)$ converges or both diverges.

Since $\sum_{n=1}^{\infty}\left(\frac{1}{n^{1/2}}\right)$ diverges, $\sum_{n=1}^{\infty}\left(\frac{n}{\sqrt{n^3+3n}}\right)$ also diverges.

51) $\sum_{n=1}^{\infty}\left(\frac{1}{\sqrt{n^3+2n}}\right)$. Explain why this series converges. Use Limit Comparison Test.

Explanation:

Let $a_n = \frac{1}{\sqrt{n^3+2n}}$; $b_n = \frac{1}{\sqrt{n^3}} = \frac{1}{n^{3/2}}$

Note: $\sum_{n=1}^{\infty}(b_n) = \sum_{n=1}^{\infty}\left(\frac{1}{n^{3/2}}\right)$ is a p-series with p = 3/2; hence it converges.

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty}\left(\frac{a_n}{b_n}\right) = \lim_{n\to\infty}\left(\frac{\frac{1}{\sqrt{n^3+2n}}}{\frac{1}{n^{3/2}}}\right) = \lim_{n\to\infty}\left(\frac{n^{3/2}}{\sqrt{n^3+2n}}\right) = 1$$

Because $\lim_{n\to\infty}\left(\frac{a_n}{b_n}\right) = 1 > 0$ Limit Comparison Test says that either

both $\sum_{n=2}^{\infty}(a_n)$ and $\sum_{n=2}^{\infty}(b_n)$ converges or both diverges.

Since $\sum_{n=1}^{\infty}(b_n) = \sum_{n=1}^{\infty}\left(\frac{1}{n^{3/2}}\right)$ converges, $\sum_{n=1}^{\infty}\left(\frac{1}{\sqrt{n^3+2n}}\right)$ also converges.

52) $\sum_{n=1}^{\infty}\left(\frac{n+1}{n(n+2)}\right)$. Explain why this series diverges. Use Limit Comparison Test.

Explanation:

Let $a_n = \frac{n+1}{n(n+2)}$; $b_n = \frac{n}{n(n)} = \frac{1}{n}$

Note: $\sum_{n=1}^{\infty}(b_n) = \sum_{n=1}^{\infty}\left(\frac{1}{n}\right)$ is a p-series with p = 1; hence it diverges.

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty}\left(\frac{a_n}{b_n}\right) = \lim_{n\to\infty}\left(\frac{\frac{n+1}{n(n+2)}}{\frac{1}{n}}\right) = \lim_{n\to\infty}\left(\frac{n+1}{n+2}\right) = 1$$

Because $\lim_{n\to\infty}\left(\frac{a_n}{b_n}\right) = 1 > 0$ Limit Comparison Test says that either

both $\sum_{n=2}^{\infty}(a_n)$ and $\sum_{n=2}^{\infty}(b_n)$ converges or both diverges.

Since $\sum_{n=1}^{\infty}(b_n) = \sum_{n=1}^{\infty}\left(\frac{1}{n}\right)$ diverges, $\sum_{n=1}^{\infty}\left(\frac{n+1}{n(n+2)}\right)$ also diverges.

54) $\sum_{n=1}^{\infty}\left(\frac{1}{3^n-5}\right)$. Explain why this series converges. Use Limit Comparison Test.

Explanation:

Let $a_n = \frac{1}{3^n-5}$; $b_n = \frac{1}{3^n}$

Note: $\sum_{n=1}^{\infty}(b_n) = \sum_{n=1}^{\infty}\left(\frac{1}{3^n}\right) = \sum_{n=1}^{\infty}\left(\frac{1}{3}\right)^n$ is a Geometric series with r = 1/2; hence it converges.

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty}\left(\frac{a_n}{b_n}\right) = \lim_{n\to\infty}\left(\frac{\frac{1}{3^n-5}}{\frac{1}{3^n}}\right) = \lim_{n\to\infty}\left(\frac{3^n}{3^n-5}\right) = 1$$

Because $\lim_{n\to\infty}\left(\frac{a_n}{b_n}\right) = 1 > 0$ Limit Comparison Test says that either

both $\sum_{n=2}^{\infty}(a_n)$ and $\sum_{n=2}^{\infty}(b_n)$ converges or both diverges.

Since $\sum_{n=1}^{\infty}(b_n) = \sum_{n=1}^{\infty}\left(\frac{1}{3^n}\right) = \sum_{n=1}^{\infty}\left(\frac{1}{3}\right)^n$ converges, $\sum_{n=1}^{\infty}\left(\frac{1}{3^n-5}\right)$ also converges.

55) $\sum_{n=1}^{\infty}\left(\frac{(-1)^n}{n^5}\right)$. Explain why this series converges. Use Alternating Series Test.

Explanation:

Let $a_n = \frac{1}{n^5}$.

a) Show $\lim_{n\to\infty} a_n = 0$: $\lim_{n\to\infty} a_n = \lim_{n\to\infty}\left(\frac{1}{n^5}\right) = 0$

b) Show a_n is decreasing.

$a_1=1$; $a_2=1/32$; $a_3 = 1/3^5$; $a_4 = 1/4^5$; $a_5 = 1/5^5$

Also, for $f(x) = \frac{1}{x^5}$, $x \geq 1$; $f'(x) = -5x^4 < 0$ for $x \geq 1$.

Hence, $a_n = \frac{1}{n^5}$ is decreasing

Therefore, by Alternating Series Test, $\sum_{n=1}^{\infty}\left(\frac{(-1)^n}{n^5}\right)$ converges.

56) $\sum_{n=1}^{\infty}\left(\frac{(-1)^n (n+1)}{n^2+1}\right)$. Explain why this series converges. Use Alternating Series Test.

Explanation:

Let $a_n = \frac{(n+1)}{n^2+1}$.

a) Show $\lim_{n\to\infty} a_n = 0$: $\quad \lim_{n\to\infty} a_n = \lim_{n\to\infty}\left(\frac{n+1}{n^2+1}\right) = \lim_{n\to\infty}\left(\frac{1}{2n}\right) = 0$

b) Show a_n is decreasing.

$a_1=1$; $a_2=3/5$; $a_3 = 4/10$; $a_4 = 5/17$; $a_5 = 6/26$

Also, for $f(x) = \frac{x+1}{x^2+1}$, $x \geq 1$; $f'(x) = \frac{(x^2+1)(1)-(x+1)(2x)}{\left(x^2+1\right)^2} = \frac{-x^2-2x+1}{\left(x^2+1\right)^2} < 0$ for $x \geq 1$.

Hence, $a_n = \frac{(n+1)}{n^2+1}$ is decreasing

Therefore, by Alternating Series Test, $\sum_{n=1}^{\infty}\left(\frac{(-1)^n (n+1)}{n^2+1}\right)$ converges.

57) $\sum_{n=2}^{\infty}\left(\frac{(-1)^n (n)}{n^2-3}\right)$. Explain why this series converges. Use Alternating Series Test.

Explanation:

Let $a_n = \frac{n}{n^3-3}$.

a) Show $\lim_{n\to\infty} a_n = 0$: $\lim_{n\to\infty} a_n = \lim_{n\to\infty}\left(\frac{n}{n^3-3}\right) = \lim_{n\to\infty}\left(\frac{1}{3n^2}\right) = 0$

b) Show a_n is decreasing.

Also, for $f(x) = \frac{x}{x^3-3}$, $x \geq 2$; $f'(x) = \frac{(x^3-3)(1)-(x)(3x^2)}{\left(x^3-3\right)^2} = \frac{-2x^3-3}{\left(x^2+1\right)^2} < 0$ for $x \geq 2$.

Hence, $a_n = \frac{n}{n^3-3}$ is decreasing

Therefore, by Alternating Series Test, $\sum_{n=2}^{\infty}\left(\frac{(-1)^n (n)}{n^2-3}\right)$ converges.

58) $\sum_{n=1}^{\infty}\left(\frac{(-1)^n\sqrt{n}}{n+1}\right)$. Explain why this series converges. Use Alternating Series Test.

Explanation:

Let $a_n = \frac{\sqrt{n}}{n+1}$.

a) Show $\lim_{n\to\infty} a_n = 0$: $\quad \lim_{n\to\infty} a_n = \lim_{n\to\infty}\left(\frac{\sqrt{n}}{n+1}\right) = \lim_{n\to\infty}\left(\frac{(1/2)n^{-1/2}}{1}\right) = \lim_{n\to\infty}\left(\frac{1}{2n^{1/2}}\right) = 0$

b) Show a_n is decreasing.

$a_1 = 1/2$; $a_2 = \frac{\sqrt{2}}{3}$; $a_3 = \frac{\sqrt{3}}{4}$; $a_4 = \frac{\sqrt{4}}{5}$;

Also, for $f(x) = \frac{\sqrt{x}}{x+1}$ $x \geq 1$;

$$f'(x) = \frac{(x+1)\left((1/2)x^{-1/2}\right) - \sqrt{x}}{(x+1)^2} = \frac{\frac{x+1}{2\sqrt{x}} - \sqrt{x}}{(x+1)^2} = \frac{\frac{(x+1)-2x}{2\sqrt{x}}}{(x+1)^2} = \frac{\frac{-x+1}{2\sqrt{x}}}{(x+1)^2} < 0 \text{ for } x \geq 2.$$

Hence, $a_n = \frac{\sqrt{n}}{n+1}$ is decreasing

Therefore, by Alternating Series Test, $\sum_{n=1}^{\infty}\left(\frac{(-1)^n\sqrt{n}}{n+1}\right)$ converges.

59) $\sum_{n=4}^{\infty}\left(\frac{(-1)^n n}{n-3}\right)$. Explain why this series diverges.

Explanation:

Let $a_n = \frac{n}{n-3}$.

$a)$ Show $\lim_{n\to\infty} a_n = 0$: $\lim_{n\to\infty} a_n = \lim_{n\to\infty}\left(\frac{n}{n-3}\right) = \lim_{n\to\infty}\left(\frac{1}{1}\right) = 1$

Therefore, $\sum_{n=4}^{\infty}\left(\frac{(-1)^n n}{n-3}\right)$ diverges by nth Term Test.

60) $\sum_{n=2}^{\infty}\left(\frac{(-1)^n \ln n^3}{n}\right)$. Explain why this series converges.

Explanation:

Let $a_n = \frac{\ln n^3}{n}$.

$a)$ Show $\lim_{n\to\infty} a_n = 0$: $\lim_{n\to\infty} a_n = \lim_{n\to\infty}\left(\frac{\ln n^3}{n}\right) = \lim_{n\to\infty}\left(\frac{3\ln n}{n}\right) = \lim_{n\to\infty}\left(\frac{3(1/n)}{1}\right) = \lim_{n\to\infty}\left(\frac{3}{n}\right) = 0$

$b)$ Let $f(x) = \frac{\ln x^3}{x} = \frac{3\ln x}{x}$; $f'(x) = \frac{(x)(3/x) - (3\ln x)(1)}{x^2} = \frac{3 - 3\ln x}{x^2} < 0$ for $x \geq 3$

Hence, $a_n = \frac{\ln n^3}{n}$ is decreasing.

Therefore, by Alternating Series Test, $\sum_{n=2}^{\infty}\left(\frac{(-1)^n \ln n^3}{n}\right)$ converges.

61) $\sum_{n=1}^{\infty}\left(\frac{3n-1}{2n+5}\right)^{n}$.   Explain why this series diverges.

Explanation:

Let $a_n = \left(\frac{3n-1}{2n+5}\right)^{n}$. $\sqrt[n]{a_n} = \sqrt[n]{\left(\frac{3n-1}{2n+5}\right)^{n}} = \frac{3n-1}{2n+5}$

$$\lim_{n\to\infty}\sqrt[n]{a_n} = \lim_{n\to\infty}\left(\frac{3n-1}{2n+5}\right) = \lim_{n\to\infty}\left(\frac{3}{2}\right) = \frac{3}{2} > 1$$

Therefore, $\sum_{n=1}^{\infty}\left(\frac{3n-1}{2n+5}\right)$ diverges by Root Test.

62) $\sum_{n=1}^{\infty}\left(\frac{4n}{7n-1}\right)^{n}$.   Explain why this series converges.

Explanation:

Let $a_n = \left(\frac{4n}{7n-1}\right)^{n}$. $\sqrt[n]{a_n} = \sqrt[n]{\left(\frac{4n}{7n-1}\right)^{n}} = \frac{4n}{7n-1}$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty}\left(\frac{4n}{7n-1}\right) = \lim_{n\to\infty}\left(\frac{4}{7}\right) = \frac{4}{7}$$

Therefore, $\sum_{n=1}^{\infty}\left(\frac{4n}{7n-1}\right)^{n}$ converges by Root Test.

63) $\sum_{n=1}^{\infty}\left(\frac{n}{e^{n^2}}\right)$. Explain why this series converges.

Explanation:

Let $a_n = \frac{n}{e^{n^2}}$; $a_{n+1} = \frac{n+1}{e^{(n+1)^2}} = \frac{n+1}{e^{n^2+2n+1}} = \frac{n+1}{e^{n^2}e^{2n+1}}$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{n+1}{e^{n^2}e^{2n+1}}\cdot\frac{e^{n^2}}{n}\right| = \lim_{n\to\infty}\left(\frac{n+1}{n}\cdot\frac{1}{e^{2n+1}}\right) = \lim_{n\to\infty}\left(\frac{n+1}{n}\right)\cdot\lim_{n\to\infty}\left(\frac{1}{e^{2n+1}}\right) = (1)(0) = 0$$

Therefore, $\sum_{n=1}^{\infty}\left(\frac{n}{e^{n^2}}\right)$ converges by Root Test.

65) $\sum_{n=1}^{\infty}\left(\frac{2^n}{n^3}\right)$. Explain why this series diverges.

Explanation:

Let $a_n = \frac{2^n}{n^3}$; $a_{n+1} = \frac{2^{n+1}}{(n+1)^3} = \frac{2^n 2^1}{(n+1)^3}$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{2^n 2^1}{(n+1)^3}\cdot\frac{n^3}{2^n}\right| = \lim_{n\to\infty}\left(\frac{2^1 n^3}{(n+1)^3}\right) = \lim_{n\to\infty}\frac{2n^3}{(n+1)^3} = 2$$

Therefore, $\sum_{n=1}^{\infty}\left(\frac{2^n}{n^3}\right)$ diverges by Ratio Test.

64) $\sum_{n=1}^{\infty}\left(\frac{n!}{e^n}\right)$. Explain why this series diverges.

Explanation:

Let $a_n = \frac{n!}{e^n}$; $a_{n+1} = \frac{(n+1)!}{e^{n+1}} = \frac{(n+1)\left[(n)(n-1)\cdots(3)(2)(1)\right]}{e^n e^1} = \frac{(n+1)[n!]}{e^n e^1}$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)[n!]}{e^n e^1}\cdot\frac{e^n}{n!}\right| = \lim_{n\to\infty}\left((n+1)\cdot\frac{1}{e^1}\right) = \lim_{n\to\infty}(n+1)\cdot\lim_{n\to\infty}\left(\frac{1}{e}\right) = (\infty)\left(\frac{1}{e}\right) = \infty$$

Therefore, $\sum_{n=1}^{\infty}\left(\frac{n!}{e^n}\right)$ diverges by Ratio Test.

67) $\sum_{n=1}^{\infty} n\left(\frac{3}{5}\right)^n$

For $n=5$, $S_5 = \sum_{n=1}^{5}(n)\left(\frac{3}{5}\right)^n = (1)\left(\frac{3}{5}\right)^1 + (2)\left(\frac{3}{5}\right)^2 + (3)\left(\frac{3}{5}\right)^3 + (4)\left(\frac{3}{5}\right)^4 + (5)\left(\frac{3}{5}\right)^5$

For $n=10$, $S_{10} = \sum_{n=1}^{10}(n)\left(\frac{3}{5}\right)^n = (1)\left(\frac{3}{5}\right)^1 + (2)\left(\frac{3}{5}\right)^2 + \cdots + (9)\left(\frac{3}{5}\right)^9 + (10)\left(\frac{3}{5}\right)^5$

For $n=15$, $S_{15} = \sum_{n=1}^{15}(n)\left(\frac{3}{5}\right)^n = (1)\left(\frac{3}{5}\right)^1 + (2)\left(\frac{3}{5}\right)^2 + \cdots + (9)\left(\frac{3}{5}\right)^{14} + (10)\left(\frac{3}{5}\right)^{15}$

For $n=20$, $S_{20} = \sum_{n=1}^{20}(n)\left(\frac{3}{5}\right)^n = (1)\left(\frac{3}{5}\right)^1 + (2)\left(\frac{3}{5}\right)^2 + \cdots + (9)\left(\frac{3}{5}\right)^{19} + (10)\left(\frac{3}{5}\right)^{20}$

For $n=25$, $S_{25} = \sum_{n=1}^{25}(n)\left(\frac{3}{5}\right)^n = (1)\left(\frac{3}{5}\right)^1 + (2)\left(\frac{3}{5}\right)^2 + \cdots + (9)\left(\frac{3}{5}\right)^{24} + (10)\left(\frac{3}{5}\right)^{25}$

69) $f(x) = e^{-2x}$. Write a 3rd degree polynomial to approximate $f(x)$ values around $c = 0$.

$f(x) = e^{-2x}$ $\qquad f(0) = e^0 = 1$

$f'(x) = -2e^{-2x}$ $\qquad f'(0) = -2e^0$ = -2

$f''(x) = 4e^{-2x}$ $\qquad f''(0) = 4e^0$ = 4

$f'''(x) = -8e^{-2x}$ $\qquad f'''(0) = -8e^0$ = -8

$$P_3(x) = f(c) + f'(c)(x-c) + \frac{f''(c)(x-c)^2}{2!} + \frac{f'''(c)(x-c)^3}{3!}$$

$$P_3(x) = 1 + (-2)(x) + \frac{(4)(x)^2}{2} + \frac{(-8)(x)^3}{6}$$

70) $f(x) = \cos \pi x$. Write a 4th degree polynomial to approximate $f(x)$ values around $c = 0$.

$f(x) = \cos \pi x$ $\qquad f(0) = 1$

$f'(x) = -\pi \sin \pi x$ $\qquad f'(0) = 0$

$f''(x) = -\pi^2 \cos \pi x$ $\qquad f''(0) = -\pi^2$

$f'''(x) = \pi^3 \sin \pi x$ $\qquad f'''(0) = 0$

$f^{(4)}(x) = \pi^4 \cos \pi x$ $\qquad f'''(0) = \pi^4$

$$P_4(x) = f(c) + f'(c)(x-c) + \frac{f''(c)(x-c)^2}{2!} + \frac{f'''(c)(x-c)^3}{3!} + \frac{f^{(4)}(c)(x-c)^4}{4!}$$

$$P_4(x) = 1 + (0)(x) + (0)(x) + \frac{(-\pi^2)(x)^2}{2} + \frac{(0)(x)^3}{6} + \frac{(\pi^4)(x)^4}{24}$$

71) $f(x)=e^{-3x}$. Write a 3rd degree polynomial to approximate $f(x)$ values around $c=0$.

$f(x)=e^{-3x}$ $\qquad$ $f(0)=e^{0}=1$

$f'(x)=-3e^{-3x}$ $\qquad$ $f'(0)=-3e^{0}$ = -3

$f''(x)=9e^{-3x}$ $\qquad$ $f''(0)=9e^{0}$ = 9

$f'''(x)=-27e^{-3x}$ $\qquad$ $f'''(0)=-27e^{0}$ = -27

$$P_3(x)=f(c)+f'(c)(x-c)+\frac{f''(c)(x-c)^2}{2!}+\frac{f'''(c)(x-c)^3}{3!}$$

$$P_3(x)=1+(-3)(x)+\frac{(9)(x)^2}{2}+\frac{(-27)(x)^3}{6}$$

72) $f(x)=\tan x$. Write a 3rd degree polynomial to approximate $f(x)$ values around $c=-\frac{\pi}{4}$.

$f(x)=\tan x$ $\qquad$ $f\left(-\frac{\pi}{4}\right)=-1$

$f'(x)=\sec^2 x$ $\qquad$ $f'\left(-\frac{\pi}{4}\right)=2$

$f''(x)=2\sec^2 x\tan x$ $\qquad$ $f''\left(-\frac{\pi}{4}\right)=-4$

$f'''(x)=4\sec^2 x\tan^2 x+2\sec^4 x$ $\qquad$ $f'''\left(-\frac{\pi}{4}\right)=16$

$$P_3(x)=f(c)+f'(c)(x-c)+\frac{f''(c)(x-c)^2}{2!}+\frac{f'''(c)(x-c)^3}{3!}$$

$$P_3(x)=1+(2)(x--\frac{\pi}{4})+\frac{(-4)(x--\frac{\pi}{4})^2}{2}+\frac{(16)(x--\frac{\pi}{4})^3}{6}$$

$$P_3(x)=1+(2)\left(x+\frac{\pi}{4}\right)+\frac{(-4)\left(x+\frac{\pi}{4}\right)^2}{2}+\frac{(16)\left(x+\frac{\pi}{4}\right)^3}{6}$$

75) $\sum_{n=0}^{\infty}\left(\frac{x}{10}\right)^{n}$.   Find interval of convergence for this series.

Let $u_n = \left(\frac{x}{10}\right)^{n}$ and $u_{n+1} = \left(\frac{x}{10}\right)^{n+1} = \left(\frac{x}{10}\right)^{n}\left(\frac{x}{10}\right)^{1}$

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty} a_n \left|\frac{\left(\frac{x}{10}\right)^{n}\left(\frac{x}{10}\right)^{1}}{\left(\frac{x}{10}\right)^{n}}\right| = \left|\left(\frac{x}{10}\right)\right|$$

Set $\left|\left(\frac{x}{10}\right)\right| < 1.$

Hence, $-1 < \frac{x}{10} < 1 \quad \Leftrightarrow \quad -10 < x < 10$

Therefore, interval of convergence is (-10, 10).

77) $\sum_{n=0}^{\infty} \frac{(-1)^n (x-2)^n}{(n+1)^2}$. Find interval of convergence for this series.

Let $u_n = \frac{(-1)^n (x-2)^n}{(n+1)^2}$ and $u_{n+1} = \frac{(-1)^{n+1} (x-2)^{n+1}}{((n+1)+1)^2} = \frac{(-1)^{n+1} (x-2)^n (x-2)^1}{(n+2)^2}$

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{(-1)^{n+1}(x-2)^n(x-2)^1}{(n+2)^2}\cdot\frac{(n+1)^2}{(-1)^n(x-2)^n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)^2}{(n+2)^2}\cdot(x-2)^1\right|$$

$$= \lim_{n\to\infty}\left|\frac{(n+1)^2}{(n+2)^2}\right|\cdot\lim_{n\to\infty}|x-2| = 1\cdot|x-2| = |x-2|$$

Set $|x-2| < 1$.

Hence, $-1 < x-2 < 1 \iff 1 < x < 3$

Note when $x=1$, $\sum_{n=0}^{\infty}\frac{(-1)^n(x-2)^n}{(n+1)^2} = \sum_{n=0}^{\infty}\frac{(-1)^n(-1)^n}{(n+1)^2} = \sum_{n=0}^{\infty}\frac{1}{(n+1)^2}$ converges by Integral Test.

Note when $x=3$, $\sum_{n=0}^{\infty}\frac{(-1)^n(x-2)^n}{(n+1)^2} = \sum_{n=0}^{\infty}\frac{(-1)^n(1)^n}{(n+1)^2} = \sum_{n=0}^{\infty}\frac{(-1)^n}{(n+1)^2}$ converges by Alternating Series Test.

Therefore, interval of convergence is [1, 3].

79) $\sum_{n=0}^{\infty} n!(x-2)^n$.   Find interval of convergence for this series.

Let $u_n = n!(x-2)^n$ and $u_{n+1} = (n+1)!(x-2)^{n+1} = (n+1)(n)!(x-2)^n(x-2)^1$

Note that $(n+1)! = (n+1)\cdot\left[(n)(n-1)(n-2)\cdots(3)(2)(1)\right] = (n+1)(n)!$

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)(n)!(x-2)^n(x-2)^1}{n!(x-2)^n}\right| = \lim_{n\to\infty}\left|(n+1)(x-2)\right|$$

$$= \lim_{n\to\infty}\left|(n+1)\right|\cdot\lim_{n\to\infty}\left|(x-2)\right| = \infty\cdot\lim_{n\to\infty}\left|(x-2)\right| = \infty$$

Therefore, $\sum_{n=0}^{\infty} n!(x-2)^n$ only converges when $x=2$.

When $x=2$, $\sum_{n=0}^{\infty} n!(x-2)^n = \sum_{n=0}^{\infty} n!(0)^n = \sum_{n=0}^{\infty} 0 = 0$.

85) $f(x) = \dfrac{2}{3-x}$.   Find a power series representation for $f(x)$ centered at $c=0$.

Note: $\dfrac{2}{3-x} = \dfrac{2/3}{3/3 - x/3} = \dfrac{2/3}{1-x/3} = \dfrac{a}{1-r}$

So power series representation for $f(x)$ is $\sum_{n=0}^{\infty} ar^n = \sum_{n=0}^{\infty}\left(\dfrac{2}{3}\right)\left(\dfrac{x}{3}\right)^n$

86) $f(x) = \dfrac{3}{2+x}$.   Find a power series representation for $f(x)$ centered at $c=0$.

Note: $\dfrac{3}{2+x} = \dfrac{3/2}{2/2-(-x)} = \dfrac{3/2}{1-(-x)} = \dfrac{a}{1-r}$

So power series representation for $f(x)$ is $\sum_{n=0}^{\infty} ar^n = \sum_{n=0}^{\infty}\left(\dfrac{3}{2}\right)(-x)^n$

87) $f(x)=\dfrac{6}{4-x}$. Find a power series representation for $f(x)$ centered at $\text{c}=1$.

Note: $\dfrac{6}{4-x}=\dfrac{6}{4-(x-1)+1}=\dfrac{6}{3-(x-1)}=\dfrac{6/3}{3/3-(x-1)/3}=\dfrac{2}{1-(x-1)/3}=\dfrac{a}{1-\ r}$

So power series representation for $f(x)$ is $\sum_{\text{n=0}}^{\infty} ar^n = \sum_{\text{n=0}}^{\infty} 2\left(\dfrac{x-1}{3}\right)^n$

88) $f(x)=\dfrac{1}{3-2x}$. Find a power series representation for $f(x)$ centered at $\text{c}=0$.

Note: $\dfrac{1}{3-2x}=\dfrac{1/3}{3/3-2x/3}=\dfrac{1/3}{1-2x/3}=\dfrac{a}{1-\ r}$

So power series representation for $f(x)$ is $\sum_{\text{n=0}}^{\infty} ar^n = \sum_{\text{n=0}}^{\infty}\left(\dfrac{1}{3}\right)\left(\dfrac{2x}{3}\right)^n$