Test 2 Review

① Let $z = f(x,y) = \frac{\sqrt{x-1}}{y}$

Find domain and range of $f(x,y)$

Domain $= \{(x,y) : x-1 > 0 ; y \neq 0\}$

Range $= (-\infty, \infty)$

② Let $z = 4 - 2x + 3y$

Find level curves for $c = 0, 2, 4$

For $c=0$: $4 - 2x + 3y = 0$

For $c=2$: $4 - 2x + 3y = 2$

For $c=4$: $4 - 2x + 3y = 4$

③ Find $\lim\limits_{(x,y)\to(0,0)} \frac{x+1}{y+3}$

$$\lim_{(x,y)\to(0,0)} \frac{x+1}{y+3} = \frac{0+1}{0+3} = \frac{1}{3}$$

(4) Show that the $\lim\limits_{(x,y)\to(0,0)} \dfrac{xy}{x^2+y^2}$ does not exist.

a) Approach $(0,0)$ along the path $y=x$:

$$\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2} = \lim_{(x,y)\to(0,0)} \frac{x\cdot x}{x^2+x^2} = \frac{1}{2}$$

b) Approach $(0,0)$ along the path $y=0$:

$$\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2} = \lim_{(x,y)\to(0,0)} \frac{x\cdot 0}{x^2+0^2}$$

$$= \lim_{(x,y)\to(0,0)} \frac{0}{x^2} = \text{DNE}$$

Therefore, $\lim\limits_{(x,y)\to(0,0)} \dfrac{xy}{x^2+y^2} = \text{DNE}$

(5) Find $\lim\limits_{(x,y)\to(0,0)} \dfrac{e^{-x}+e^{-y}}{x+1}$

$$\lim_{(x,y)\to(0,0)} \frac{e^{-x}+e^{-y}}{x+1} = \frac{e^0+e^0}{0+1} = \frac{2}{1} = 2$$

⑥ Let $f(x,y) = 5x^2 + 8x + 4y$
Find f_x and f_y

$f_x = 10x + 8$; $f_y = 4$

⑦ Let $f(x,y) = e^x \cdot \sin y$.
Find f_x and f_y

$f_x = e^x \cdot \sin y$; $f_y = e^x \cdot \cos y$

⑧ Let $f(x,y,z) = 2yx^2 + 8xyz - 8xz^2$
Find f_x, f_y, and f_z

$f_x = 2y(2x) + 8yz - 8z^2$
$= 4xy + 8yz - 8z^2$

$f_y = 2x^2 + 8xz - 0 = 2x^2 + 8xz$

$f_z = 0 + 8xy - 8x(2z) = 8xy - 16xz$

⑨ Let $f(x,y) = x^2 \cdot e^{5y}$
Find f_x and f_y

$f_x = e^{5y} \cdot (2x) = 2x e^{5y}$

$f_y = x^2 \cdot e^{5y} \cdot 5 = 5x^2 e^{5y}$

(10) $f(x,y) = 3x^2 - xy + 2y^3$

$f_x = 6x - y + 0 = 6x - y$

$f_{xx} = 6 - 0 = 6$

$f_{xy} = 0 - 1 = -1$

$f_y = 0 - x + 6y^2 = -x + 6y^2$

$f_{yy} = 0 + 12y = 12y$

$f_{yx} = -1$

(11) Let $w = \frac{4xy}{z} = \frac{4}{z} xy = 4x \cdot y \cdot z^{-1}$

Let $x = 2r + t$; $y = rt$; $z = 2r - t$

Find $\frac{\partial w}{\partial r}$ and $\frac{\partial w}{\partial t}$

$$\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x} \cdot \frac{\partial x}{\partial r} + \frac{\partial w}{\partial y} \cdot \frac{\partial y}{\partial r} + \frac{\partial w}{\partial z} \cdot \frac{\partial z}{\partial r}$$

$$= \left(\frac{4y}{z}\right)(2) + \left(\frac{4x}{z}\right)(t) + \left(-4xyz^{-2}\right)(2)$$

$$\frac{\partial w}{\partial t} = \frac{\partial w}{\partial x} \cdot \frac{\partial x}{\partial t} + \frac{\partial w}{\partial y} \cdot \frac{\partial y}{\partial t} + \frac{\partial w}{\partial z} \cdot \frac{\partial z}{\partial t}$$

$$= \left(\frac{4y}{z}\right)(1) + \left(\frac{4x}{z}\right)(r) + \left(-4xyz^{-2}\right)(-1)$$

(12) Let $w = \ln(x + y^2)$

$x = 2t$

$y = 4 - 3t$

Find dw/dt

$$\frac{dw}{dt} = \left(\frac{\partial w}{\partial x}\right)\left(\frac{dx}{dt}\right) + \left(\frac{\partial w}{\partial y}\right)\left(\frac{dy}{dt}\right) \quad \text{Chain Rule}$$

$$= \left(\frac{1}{x+y^2}\right)(2) + \left(\frac{1}{x+y^2} \cdot 2y\right)(-3)$$

$$= \frac{2}{x+y^2} - \frac{6y}{x+y^2}$$

(13) Let $f(x,y) = x^2 + y^2$ $\quad P(-5,5)$

Find directional derivative of $f(x,y)$ at $P(-5,5)$ in the direction of $\underset{\sim}{v} = \langle 3, 4\rangle$

Let $\underset{\sim}{u} = \frac{\underset{\sim}{v}}{\|\underset{\sim}{v}\|} = \frac{\langle 3, 4\rangle}{5} = \langle 3/5, 4/5\rangle$

$\nabla f = \langle f_x, f_y\rangle = \langle 2x, 2y\rangle$

$\nabla f(-5,5) = \langle 2(-5), 2(5)\rangle = \langle -10, 10\rangle$

$D_{\underset{\sim}{u}} f(x,y) = \nabla f \cdot \underset{\sim}{u} = \langle -10, 10\rangle \cdot \langle 3/5, 4/5\rangle$

$= 2$

(14) Let $w = x^2 + yz$; $P(1, 2, 3)$

Find directional derivative of w at P in the direction of $\underset{\sim}{v} = \langle -1, 2, 4 \rangle$

Let $\underset{\sim}{u} = \dfrac{\underset{\sim}{v}}{\|\underset{\sim}{v}\|} = \dfrac{\langle -1, 2, 4 \rangle}{\sqrt{21}} = \left\langle \dfrac{-1}{\sqrt{21}}, \dfrac{2}{\sqrt{21}}, \dfrac{4}{\sqrt{21}} \right\rangle$

$\nabla w = \langle w_x, w_y, w_z \rangle = \langle 2x, z, y \rangle$

$D_{\underset{\sim}{u}} w(x, y, z) = \nabla w \cdot \underset{\sim}{u} = \langle 2x, z, y \rangle \cdot \underset{\sim}{u}$

At $P(1, 2, 3)$

$D_{\underset{\sim}{u}} w(x, y, z) = \langle 2(1), 3, 2 \rangle \langle -1/\sqrt{21}, 2/\sqrt{21}, \frac{4}{\sqrt{21}} \rangle$

$= 2.618614$

(15) Let $z = x^2 y$; $P(2, 5)$

Find maximum directional derivative at P.
~~in the direction~~

$\nabla z = \langle z_x, z_y \rangle = \langle 2xy, x^2 \rangle$

$\nabla z(2, 5) = \langle 2(2)(5), 2^2 \rangle = \langle 20, 4 \rangle$

$\| \nabla z(2,5) \| = \| \langle 20, 4 \rangle \| = \sqrt{20^2 + 4^2} = \sqrt{416}$

Maximum directional derivative at $P = \sqrt{416}$

(16) Let $26 = x^2 + y^2 + z^2$; $P(1, 3, 4)$

Find tangent plane at P.

Let $F(x, y, z) = x^2 + y^2 + z^2 - 26 = 0$

$\nabla F = \langle F_x, F_y, F_z \rangle = \langle 2x, 2y, 2z \rangle$
$\nabla F(1, 3, 4) = \langle 2, 6, 8 \rangle$

Tangent Plane:
$$2(x-1) + 6(y-3) + 8(z-4) = 0$$

Normal Line:

$x - x_0 = at$ $\qquad$ $x - 1 = 2t$
$y - y_0 = bt$ $\qquad$ $y - 3 = 6t$
$z - z_0 = ct$ $\qquad$ $z - 4 = 8t$

(17) Let $f(x, y) = x^2 + 4y^2 + 8x - 8y + 10$

Find extremum

$f_x = 2x + 8$ $\qquad$ $f_y = 8y - 8$

Set $f_x = 0$ $\qquad$ Set $f_y = 0$
$2x + 8 = 0$ $\qquad$ $8y - 8 = 0$
$x = -4$ $\qquad$ $y = 1$

When $x = -4$, $y = 1$
$z = (-4)^2 + 4(1)^2 + 8(-4) - 8(1) + 10 = -10$

$f_{xx} = 2$ $\qquad$ $f_{xy} = 0$
$f_{yy} = 8$ $\qquad$ $d = f_{xx} \cdot f_{yy} - [f_{xy}]^2 = (2)(8) - 0 = 16 > 0$

Since $d > 0$ and $f_{xx} > 0$, $(-4, 1, -10)$ is a relative minimum.

(18) Minimize $f(x, y) = x^2 + y^2$

Constraint : $x + y = 4$

Let $g(x,y) = x + y - 4 = 0$

$\nabla f = \langle 2x, 2y \rangle$; $\nabla g = \langle 1, 1 \rangle$

Lagrange Equations : $\nabla F = \lambda \cdot \nabla g$

$$\langle 2x, 2y \rangle = \lambda \langle 1, 1 \rangle = \langle \lambda, \lambda \rangle$$

$\Rightarrow 2x = \lambda$; $2y = \lambda$

$x = \lambda/2$; $y = \lambda/2$

Constraint : $x + y = 4$

$$\lambda/2 + \lambda/2 = 4$$

$$\lambda + \lambda = 8 \quad \text{Multiply by 2}$$

$$2\lambda = 8$$

$$\lambda = 4$$

Hence , $x = \lambda/2 = 4/2 = 2$

$y = \lambda/2 = 4/2 = 2$

So $f(x, y)$ is minimized When $x = 2$, $y = 2$

(19) Company ABC produces two types of shoes (walking and running).
The cost function for producing x units of walking shoes and y units of running shoes is

$$C(x,y) = x^2 + xy + 2y^2 + 100$$

Due to budget constraint Company ABC can only produce 100 pairs of shoes.
Find x and y that will minimize $C(x,y)$.

$$\nabla C = \langle 2x + y \;,\; x + 4y \rangle = \langle C_x, C_y \rangle$$

Constraint: $x + y = 100$

Let $g(x,y) = x + y - 100 = 0$

$$\nabla g = \langle g_x, g_y \rangle = \langle 1, 1 \rangle$$

Lagrange Equs: $\nabla C = \lambda \cdot \nabla g$

$$\langle 2x + y \;,\; x + 4y \rangle = \lambda \langle 1, 1 \rangle = \langle \lambda, \lambda \rangle$$

$$\Rightarrow \underset{(1)}{2x + y = \lambda} \;;\quad \underset{(2)}{x + 4y = \lambda}$$

$$\Rightarrow \quad 2x + y = \lambda = x + 4y$$

$$2x + y = x + 4y$$

$$x = 3y$$

Constraint:

$$x + y = 100$$

$$3y + y = 100$$

$$4y = 100$$

$$y = 25 \qquad x = 75$$

(20) Evaluate $I = \int_0^{1+x} (3x + 4y)\, dy$

$$I = 3xy + 4y^2/2 = 3xy + 2y^2 \Big|_0^{1+x}$$

$$= \left[3x(1+x) + 2(1+x)^2\right] - [0 + 0]$$

$$= 3x + 3x^2 + 2(1+x)^2$$

(21) Evaluate $I = \int_0^1 \underbrace{\int_0^3 (3x + y)\, dy}_{I_2}\, dx$

$$I_2 = \int_0^3 (3x + y)\, dy = 9x + 9/2$$

$$I = \int_0^1 (9x + 9/2)\, dx = 9$$

(22) $I = \int_0^{\pi/3} \underbrace{\int_0^{\cos x} (4 + \sin x)\, dy}_{I_2}\, dx$

$$I_2 = \int_0^{\cos x} (4 + \sin x)\, dy = 4\cos x + \sin x \cos x$$

$$I = \int_0^{\pi/3} (4\cos x + \sin x \cos x)\, dx =$$

$$= 4\sin(\pi/3) + \frac{(\sin(\pi/3))^2}{2}$$