Calculus III Test 3 Review

Chapter 14

(1) $\int_0^{2x} xy^3\,dy = \frac{xy^4}{4}\Big|_0^{2x}$

$= \frac{x(2x)^4}{4} - \frac{x(0)^4}{4}$

$= 4x^5 - 0 = 4x^5$

(3) $I = \int_0^1 \underbrace{\int_0^{1+x} (3x+2y)\,dy}_{I_1}\,dx$

$I_1 = \left(3xy + 2\cdot\frac{y^2}{2}\right)\Big|_0^{1+x} =$

$= \left(3x(1+x) + (1+x)^2\right) - (0) = (4x^2+5x+1)$

$I = \int_0^1 (4x^2+5x+1)\,dx$

$= \left(\frac{4x^3}{3} + \frac{5x^2}{2} + x\right)\Big|_0^1 = 29/6$

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(5) $I = \int_0^3 \underbrace{\int_0^{\sqrt{9-x^2}} 4x \, dy}_{I_1} dx$

$I_1 = 4xy \Big|_0^{\sqrt{9-x^2}}$

$I_1 = 4x\left(\sqrt{9-x^2}\right)$

$I = \int_0^3 4x\left(\sqrt{9-x^2}\right)dx$

Let $u = 9-x^2$

$\frac{du}{dx} = -2x$

$du = -2x\,dx$

$-2du = 4x\,dx$

$I = \int \sqrt{u}\,(-2du) = -2\frac{u^{3/2}}{3/2}$

$I = \frac{-4}{3}(9-x^2)^{3/2}\Big|_0^3$

$I = 36$

(11) dydx Method:

$$\int_1^5 \int_2^4 dy\,dx = 8$$

dxdy Method:

$$\int_2^4 \int_1^5 dx\,d\cancel{x}y = 8$$

(15) $$I = \int_0^4 \underbrace{\int_0^2 4xy\,dx}_{I_1}\,dy$$

$$I_1 = \frac{4x^2y}{2}\Big|_0^2 = 8y - 0 = 8y$$

$$I = \int_0^4 8y\,dy = \frac{8y^2}{2}\Big|_0^4 = 64$$

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(19) $$V = \int_{-1}^{1} \underbrace{\int_{-1}^{1} (4 - x^2 - y^2)\, dy}_{I_1}\, dx$$

$$I_1 = \left(4 - x^2 - \frac{y^3}{3}\right)\Big|_{-1}^{1} =$$

$$= \left(4 - x^2 - \frac{1}{3}\right) - \left(4 - x^2 + \frac{1}{3}\right)$$

$$= \frac{22}{3} - 2x^2$$

$$V = \int_{-1}^{1} \left(\frac{22}{3} - 2x^2\right) dx$$

$$= \left(\frac{22}{3}x - \frac{2x^3}{3}\right)\Big|_{-1}^{1} = 40/3$$

(27) $$V = \int_0^{\pi/2} \underbrace{\int_0^3 (r\cos\theta)(r\sin\theta)^2 \, r\,dr}_{I_1}\,d\theta$$

$$I_1 = \cos\theta \cdot (\sin\theta)^2 \cdot \frac{r^5}{5} \Big|_0^3$$

$$I_1 = \cos\theta \cdot (\sin\theta)^2 \cdot \frac{243}{5} - 0$$

$$I_1 = \frac{243}{5} \cos\theta \cdot (\sin\theta)^2$$

$$V = \int_0^{\pi/2} \frac{243}{5} \cos\theta \cdot (\sin\theta)^2 \, d\theta$$

Let $u = \sin\theta$

$du = \cos\theta \cdot d\theta$

$$V = \frac{243}{5} \int^{\cancel{\pi/4}} u^2 \, du = \frac{243}{5} \frac{u^3}{3}$$

$$V = \frac{243}{15} (\sin\theta)^3 \Big|_0^{\pi/2} = \frac{243}{15}(1) - \frac{243}{15}(0)$$

$$= \frac{243}{15} = \frac{81}{5}$$

(29) $$A = \int_0^{2\pi} \underbrace{\int_0^{2+\cos\theta} r\,dr\,d\theta}_{I_1}$$

$$I_1 = \frac{r^2}{2}\Big|_0^{2+\cos\theta} = \frac{1}{2}(2+\cos\theta)^2 - 0$$

$$I_1 = \frac{1}{2}\left(4 + 4\cos\theta + \cos^2\theta\right)$$

$$I_1 = 2 + 2\cos\theta + \frac{1}{2}\cos^2\theta$$

$$A = \int_0^{2\pi}\left(2 + 2\cos\theta + \frac{1}{2}\cos^2\theta\right)d\theta$$

$$A = \left[2\theta + 2(\sin\theta) + \frac{1}{2}\left(\frac{1}{2}\theta + \frac{\sin 2\theta}{4}\right)\right]_0^{2\pi}$$

$$A = \left(4\pi + 2(0) + \frac{1}{4}(2\pi) + 0\right)$$

$$-(0 + 0 + 0 + 0)$$

$$A = 4\pi + \frac{\pi}{2} = \frac{9\pi}{2}$$

(35) $m = \int_0^2 \int_0^{x^3} kx\,dy\,dx$

$= \int_0^2 \left(kxy \Big|_0^{x^3} \right) dx$

$= \int_0^2 kx^4\,dx = \frac{kx^5}{5}\Big|_0^2 = \frac{32k}{5}$

$M_x = \int_0^2 \int_0^{x^3} kxy\,dy\,dx = 16k$

$M_y = \int_0^2 \int_0^{x^3} kx \cdot x\,dy\,dx = \frac{32k}{3}$

$\bar{x} = \frac{M_y}{m} = \frac{(32k/3)}{(32k/5)} = 5/3$

$\bar{y} = \frac{M_x}{m} = \frac{(16k)}{(32k/5)} = 5/2$

$(\bar{x}, \bar{y}) = (5/3, 5/2)$

(37) $m = \int_0^1 \int_{2x^3}^{2x} kxy \, dy\, dx$

$$m = \int_0^1 \left(kx \frac{y^2}{2} \right) \Big|_{2x^3}^{2x} dx$$

$$m = \int_0^1 \left(2kx^3 - 2kx^7\right) dx$$

$$m = k\left(\frac{2x^4}{4} - \frac{2x^8}{8}\right)\Big|_0^1 = k\left(\frac{2}{4} - \frac{2}{8}\right) = \frac{1}{4}k$$

$$m = \frac{1}{4}k$$

$$M_x = \int_0^1 \int_{2x^3}^{2x} kxy \cdot y \, dy\, dx = \frac{16k}{55}$$

$$M_y = \int_0^1 \int_{2x^3}^{2x} kxy \cdot x \, dy\, dx = \frac{8k}{45}$$

$$\bar{x} = \frac{M_y}{m} = \frac{(8k/45)}{\left(\frac{1}{4}k\right)} = \frac{32}{45}$$

$$\bar{y} = \frac{M_x}{m} = \frac{(16k/55)}{\left(\frac{1}{4}k\right)} = \frac{64}{55}$$

(39) $I_x = \iint_R y^2 \rho(x,y)\, dA$

$= \int_0^a \int_0^b kx \cdot y^2 \, dy dx$

$= \int_0^a \left(kx \cdot \frac{y^3}{3} \right)\Big|_0^b dx = \int_0^a k \cdot x \cdot \frac{b^3}{3} dx$

$= \frac{kb^3a^2}{6}$

$I_y = \iint_R x^2 \cdot \rho(x,y)\, dA = \int_0^a \int_0^b kx \cdot x^2 \, dy dx = \frac{kba^4}{4}$

$I_0 = I_x + I_y = \frac{kb^3a^2}{6} + \frac{kba^4}{4}$

$m = \int_R \int \rho(x,y)\, dA = \int_0^a \int_0^b kx \, dy dx = \frac{1}{2} kba^2$

$\bar{\bar{x}} = \sqrt{\frac{I_y}{m}} = \sqrt{\frac{kba^4/4}{kba^2/2}} = \sqrt{\frac{a^2}{2}}$

$\bar{\bar{y}} = \sqrt{\frac{I_x}{m}} = \sqrt{\frac{kb^3a^2/6}{kba^2/2}} = \sqrt{\frac{b^2}{2}}$

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(41) $f(x,y) = 25 - x^2 - y^2$

$f_x = -2x \quad ; \quad f_y = -2y$

$\sqrt{1 + (f_x)^2 + (f_y)^2} = \sqrt{1 + 4x^2 + 4y^2}$

$$S = \iint_R \sqrt{1+4x^2+4y^2}\, dA$$

Note: $r^2 = x^2 + y^2$

$$S = \int_0^{2\pi} \underbrace{\int_0^5 \sqrt{1+4r^2}\, r\,dr}_{I_1}\, d\theta$$

$$I_1 = \int_0^5 \sqrt{1+4r^2}\, r\,dr$$

Let $u = 1 + 4r^2$

$du = 8r\,dr$

$\frac{1}{8}du = r\,dr$

$$I_1 = \int \sqrt{u} \cdot \frac{1}{8} du = \frac{1}{8}\int u^{1/2} du =$$

$$= \frac{1}{8} \frac{u^{3/2}}{3/2} = \frac{1}{12}\left(1+4r^2\right)^{3/2}\Big|_0^5 = \frac{1}{12}\left(101^{3/2} - 1\right)$$

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(41) con't

$$S = \int_0^{2\pi} \left[\frac{1}{12}\left(101^{3/2} - 1\right)\right] d\theta$$

$$= \frac{1}{12}\left(101^{3/2} - 1\right)\left(2\pi - 0\right)$$

$$= \frac{\pi}{6}\left(101^{3/2} - 1\right)$$

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(47) $I = \int_0^4 \int_0^1 \underbrace{\int_0^2 (2x + y + 4z)\,dy}_{I_1}\,dz\,dx$

$$I_1 = \left(2xy + \frac{y^2}{2} + 4zy\right)\Big|_0^2 = 4x + 2 + 8z$$

$$I_2 = \int_0^1 I_1\,dz = \int_0^1 (4x + 2 + 8z)\,dz$$

$$= \left(4xz + 2z + 4z^2\right)\Big|_0^1$$

$$= 4x + 2 + 4 = 4x + 6$$

$$I = \int_0^4 I_2\,dx = \int_0^4 (4x + 6)\,dx$$

$$= \left(2x^2 + 6x\right)\Big|_0^4 = 56$$

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$$I = \int_0^a \int_0^b \underbrace{\int_0^c (x^2 + y^2 + z^2)\,dx}_{I_1}\,dy\,dz$$

$$I_1 = \left(\frac{x^3}{3} + y^2 x + z^2 x \right)\Big|_0^c$$

$$I_1 = \tfrac{1}{3}c^3 + cy^2 + cz^2$$

$$I_2 = \int_0^b I_1\,dy = \int_0^b \left(\tfrac{1}{3}c^3 + cy^2 + cz^2 \right) dy$$

$$I_2 = \tfrac{1}{3}c^3 y + \frac{cy^3}{3} + cz^2 y \Big|_0^b$$

$$I_2 = \tfrac{1}{3}bc^3 + \tfrac{1}{3}cb^3 + bcz^2$$

$$I = \int_0^a \left(\tfrac{1}{3}bc^3 + \tfrac{1}{3}cb^3 + bcz^2 \right) dz$$

$$I = \tfrac{1}{3}abc^3 + \tfrac{1}{3}ab^3c + \tfrac{1}{3}a^3bc$$

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71) $$\frac{\partial(x,y)}{\partial(u,v)} = \frac{\partial x}{\partial u}\cdot\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\cdot\frac{\partial y}{\partial u}$$

$$= \left(\tfrac{1}{2}\right)\left(-\tfrac{1}{2}\right) - \left(\tfrac{1}{2}\right)\left(\tfrac{1}{2}\right) = -\tfrac{1}{2}$$

Let $u = x + y$ ①

Let $v = x - y$ ②

(x,y)	(u,v)
$(1,2)$	$(3,-1)$
$(2,3)$	$(5,-1)$
$(3,2)$	$(5,1)$
$(2,1)$	$(3,1)$

① + ② $\Rightarrow$ $u + v = 2x$

$x = \frac{1}{2}(u+v)$

① + ② $\Rightarrow$ $u - v = 2y$

$y = \frac{1}{2}(u-v)$

$$\iint \ln(x+y)\,dA = \int_3^5 \int_{-1}^{1} \ln\left(\left[\tfrac{1}{2}(u+v)\right] + \left[\tfrac{1}{2}(u-v)\right]\right)\left|\tfrac{1}{2}\right| dv\,du$$

$$= \int_3^5 \int_{-1}^{1} \tfrac{1}{2}\ln u \; dv\,du = 2.751$$

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(73) Region R is bounded by:

$x = 1$;

$x = 4$;

$3y - x = 8 \Leftrightarrow x - 3y = -8$

$3y - x = 2 \Leftrightarrow x - 3y = -2$

Let $u = x$ (1) ; $v = x - 3y$ (2)

$v = u - 3y$

$3y = u - v$

$y = \frac{1}{3}u - \frac{1}{3}v$

$\frac{\partial x}{\partial u} = 1$ $\qquad$ $\frac{\partial y}{\partial u} = \frac{1}{3}$

$\frac{\partial x}{\partial v} = 0$ $\qquad$ $\frac{\partial y}{\partial v} = \frac{-1}{3}$

$$|J| = \frac{\partial x}{\partial u} \cdot \frac{\partial y}{\partial v} - \frac{\partial y}{\partial u} \cdot \frac{\partial x}{\partial v} = (1)\left(\frac{-1}{3}\right) - \left(\frac{1}{3}\right)(0) = \frac{-1}{3}$$

$\cancel{\iint_R (x+y)}$

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(x,y)	(u,v)
$(1,3)$	$(1,-8)$
$(4,4)$	$(4,-8)$
$(4,2)$	$(4,-2)$
$(1,1)$	$(1,-2)$

$$\iint_R (xy + x^2)\,dA$$

$$= \int_1^4 \int_{-2}^{-8} \left[(u)\left(\tfrac{1}{3}u - \tfrac{1}{3}v\right) + u^2\right]\left(-\tfrac{1}{3}\right) dv\,du$$

$$= \left(-\tfrac{1}{3}\right)\int_1^4 \int_{-2}^{-8} \left(\tfrac{4}{3}u^2 - \tfrac{1}{3}uv\right) dv\,du$$

$$= 81$$

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3) $f(x,y,z) = 2x^2 + xy + z^2$

$\nabla f = \langle f_x, f_y, f_z \rangle = \langle 4x + y, x, 2z \rangle$

7) $\underline{F}(x,y) = \langle xy^2 - x^2, x^2y + y^2 \rangle$

let $M = xy^2 - x^2$; $N = x^2y + y^2$

$\frac{\partial M}{\partial y} = 2xy$; $\frac{\partial N}{\partial x} = 2xy$

$\underline{F}$ is conservative

Let $\nabla f = \underline{F} = \langle xy^2 - x^2, x^2y + y^2 \rangle = \langle f_x, f_y \rangle$

Find potential function $f(x,y)$:

$f(x,y) = \int f_x \, dx = \int (xy^2 - x^2) dx =$

$= \frac{x^2}{2} y^2 - \frac{x^3}{3} + C(x)$

$f(x,y) = \int f_y dy = \int (x^2y + y^2) dy$

$= (x^2y^2/2 + y^3/3) + C(y)$

Potential function $f(x,y) = \frac{1}{2} x^2y^2 + \frac{y^3}{3} - \frac{x^3}{3} + C$

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(9) $\mathbf{F} = \langle 4xy^2, \; 2x^2, \; 2z \rangle = \langle M, N, P \rangle$

$\frac{\partial M}{\partial y} = 8xy \; ; \; \frac{\partial N}{\partial x} = 4x$

So $\mathbf{F}$ is not conservative.

(13) $\mathbf{F} = \langle x^2, \; xy^2, \; x^2 z \rangle = \langle M, N, P \rangle$

$\text{div } \mathbf{F} = \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} + \frac{\partial P}{\partial z}$

$= 2x + 2xy + x^2$

$\text{Curl } \mathbf{F} = \left(\frac{\partial P}{\partial y} - \frac{\partial N}{\partial z} \right) \mathbf{i} - \left(\frac{\partial P}{\partial x} - \frac{\partial M}{\partial z} \right) \mathbf{j}$

$+ \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) \mathbf{k}$

$= \langle 0, -2xz, \; y^2 \rangle$

#15 $\underset{\sim}{F} = \langle \cos y + y\cos x,\ \sin x - x\sin y,\ xyz \rangle$

$= \langle M, N, P \rangle$

$$\operatorname{div} \underset{\sim}{F} = \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} + \frac{\partial P}{\partial z}.$$

$$= -y\sin x - x\cos y + xy$$

$$\operatorname{Curl} \underset{\sim}{F} = \left(\frac{\partial P}{\partial y} - \frac{\partial N}{\partial z}\right)\underset{\sim}{i} - \left(\frac{\partial P}{\partial x} - \frac{\partial M}{\partial z}\right)\underset{\sim}{j}$$

$$+ \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)\underset{\sim}{k}$$

$$= \langle xz, -yz, 0 \rangle$$

#21 Let $x = 3t$; $y = 4t$; $ds = \sqrt{9+16} = 5dt$
$0 \le t \le 1$

a) $\int_C (x^2+y^2)\,ds$

$= \int_0^1 (9t^2 + 16t^2) \cdot 5dt$

$= 125\int_0^1 (t^2)\,dt = 125 \frac{t^3}{3}\Big|_0^1 = 125/3$

b) Let $x = \cos t$; $y = \sin t$; $0 \le t \le 2\pi$

Note : $\cos^2 t + \sin^2 t = 1$; $ds = \sqrt{\cos^2 t + \sin^2 t} = 1dt$

$\int_C (x^2+y^2)\,ds = \int_0^{2\pi} (\cos^2 t + \sin^2 t)\,dt$

$= \int_0^{2\pi} 1dt = t\Big|_0^{2\pi} = 2\pi$

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23 $\mathbf{r} = \langle 1-\sin t,\ 1-\cos t \rangle \qquad 0 \le t \le 2\pi$

Let $x = 1-\sin t \qquad y = 1-\cos t$

$\frac{dx}{dt} = -\cos t \qquad \frac{dy}{dt} = \sin t$

$ds = \sqrt{(-\cos t)^2 + (\sin t)^2}\ dt = \sqrt{1}\ dt = dt$

$$\int_C (x^2+y^2)\,ds = \int_0^{2\pi} \left[(1-\sin t)^2 + (1-\cos t)^2\right](1)\,dt$$

$$= \int (1 - 2\sin t + \sin^2 t + 1 - 2\cos t + \cos^2 t)\,dt$$

$$= \int [3 - 2\sin t - 2\cos t]\,dt$$

$$= \Big[3t + 2\cos t - 2\sin t\Big]_0^{2\pi} = 6\pi$$

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(31) $\underset{\sim}{F} = \langle xy, \ 2xy \rangle$

$\underset{\sim}{r}(t) = \langle t^2, \ t^2 \rangle = \langle x, y \rangle \quad 0 \le t \le 1$

$\dfrac{d\underset{\sim}{r}}{dt} = r'(t) = \langle 2t, \ 2t \rangle$

$\underset{\sim}{F}(t) = \langle t^2(t^2), \ 2(t^2)(t^2) \rangle = \langle t^4, 2t^4 \rangle$

$\displaystyle\int_C \underset{\sim}{F} \cdot d\underset{\sim}{r} = \int_0^1 \langle t^4, 2t^4 \rangle \langle 2t, 2t \rangle \, dt$

$\displaystyle = \int (2t^5 + 4t^5) dt$

$\displaystyle = \int 6t^5 dt = \frac{6t^6}{6}\Big|_0^1 =$

$= \underline{1}$

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33. $\underset{\sim}{F} = \langle x, y, z \rangle$

$C: \underset{\sim}{r}(t) = \langle 2\cos t, 2\sin t, t \rangle \quad 0 \le t \le 2\pi$

$= \langle x, y, z \rangle$

$F(t) = \langle 2\cos t, 2\sin t, t \rangle$

$\frac{d\underset{\sim}{r}}{dt} = \underset{\sim}{r}'(t) = \langle -2\sin t, 2\cos t, 1 \rangle$

$$\int_C \underset{\sim}{F} \cdot d\underset{\sim}{r} = \int_0^{2\pi} \langle 2\cos t, 2\sin t, t \rangle \langle -2\sin t, 2\cos t, 1 \rangle dt$$

$$= \int (-4\cos t \sin t + 4 \sin t \cos t + t) dt$$

$$= \int t \, dt = \frac{t^2}{2}\Big|_0^{2\pi} = 2\pi^2$$