Summary of Vectors in Space

Vector Operations:

Let $\mathbf{u} = \langle u_1, u_2 \rangle$ and $\mathbf{v} = \langle v_1, v_2 \rangle$

a) $\mathbf{u} + \mathbf{v} = \langle u_1, u_2 \rangle + \langle v_1, v_2 \rangle = \langle u_1 + v_1, u_2 + v_2 \rangle$

b) $\text{-}\mathbf{u} = \langle -u_1, -u_2 \rangle$; $\quad -\mathbf{v} = \langle -v_1, -v_2 \rangle$

c) $\mathbf{u} - \mathbf{v} = \mathbf{u} + (-\mathbf{v}) = \langle u_1, u_2 \rangle + \langle -v_1, -v_2 \rangle = \langle u_1 + -v_1, u_2 + -v_2 \rangle$

d) Scalar Multiple:

$$b\mathbf{u} = b\langle u_1, u_2 \rangle = \langle bu_1, bu_2 \rangle$$

$$c\mathbf{v} = c\langle v_1, v_2 \rangle = \langle cv_1, cv_2 \rangle$$

Distance Between Points:

Distance Between the points (x_1, y_1, z_1) and (x_2, y_2, z_2):

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

Equation of Sphere

Let (x_0, y_0, z_0) be center of sphere and r be radius of sphere.

Then equation of sphere: $(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = r^2$

Dot Product of Two Vectors

Let $\mathbf{u} = \langle u_1, u_2 \rangle$ and $\mathbf{v} = \langle v_1, v_2 \rangle$

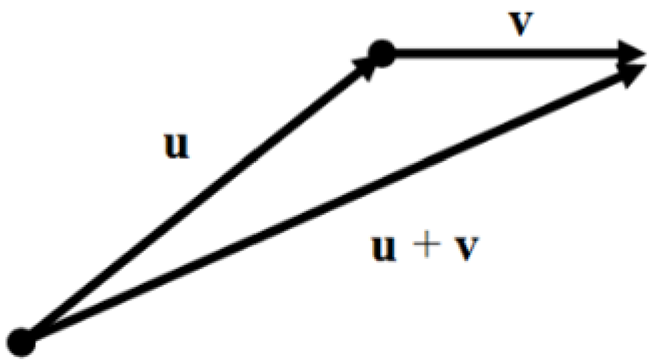
Then: $\mathbf{u} \cdot \mathbf{v} = u_1 \cdot v_1 + u_2 v_2$

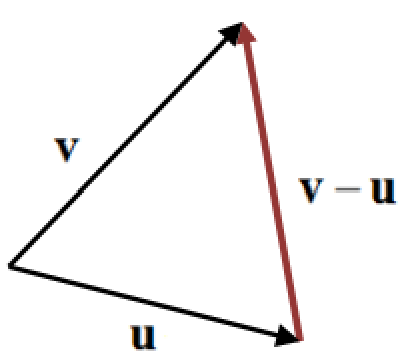
Angle Between Two Vectors

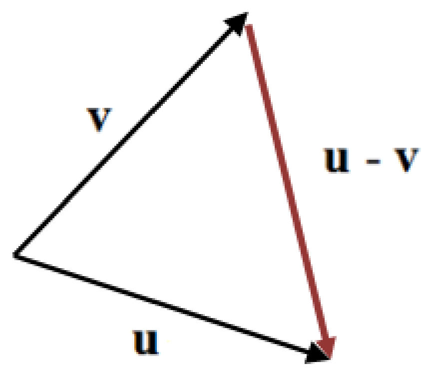
Let θ be the angle between $\mathbf{u}$ and $\mathbf{v}$, where $0 \le \theta \le \pi$.

Then $\mathbf{u} \cdot \mathbf{v} = \|\mathbf{u}\| \cdot \|\mathbf{v}\| \cos\theta$ or $\cos\theta = \dfrac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\| \cdot \|\mathbf{v}\|}$

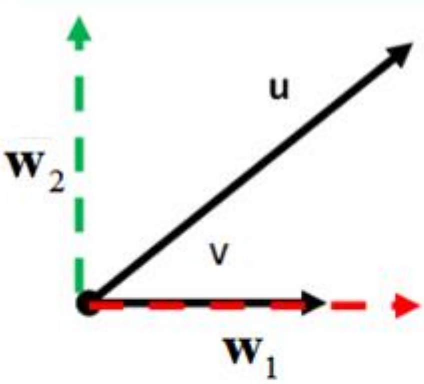
Vector Addition and Subtraction







Projection and Vector Components



Note: $\mathbf{u} = \mathbf{w}_1 + \mathbf{w}_2$

$\mathbf{w}_1$ = projection of $\mathbf{u}$ onto v = $\text{proj}_{\mathbf{v}}\mathbf{u}$.

$$\mathbf{w}_1 = \text{proj}_{\mathbf{v}}\mathbf{u} = \left(\frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{v}\|^2}\right)\mathbf{v}$$

$\mathbf{w}_2 = \mathbf{u} - \mathbf{w}_2$ = vector component of $\mathbf{u}$ orthogonal to $\mathbf{v}$

Cross Product fo Two Space Vectors in Space

Let $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$ and $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$$

$$= (u_2 v_3 - u_3 v_2)\mathbf{i} - (u_1 v_3 - u_3 v_1)\mathbf{j} + (u_1 v_2 - u_2 v_1)\mathbf{k}$$

and

$$\|\mathbf{u} \times \mathbf{v}\| = \sqrt{(u_2 v_3 - u_3 v_2)^2 + (u_1 v_3 - u_3 v_1)^2 + (u_1 v_2 - u_2 v_1)^2}$$

$(\mathbf{u} \times \mathbf{v})$ and $\mathbf{u}$ are orthogonal.

$(\mathbf{u} \times \mathbf{v})$ and $\mathbf{v}$ are orthogonal.

Lines in space

Line L is represented by the parametric equations:

$$x - x_1 = ta, \quad y - y_1 = tb, \quad z - z_1 = tc$$

Planes in space

Equation of plane: $a(x - x_1) + b(y - y_1) + c(z - z_1) = 0$

Distance between point and plane

$$\text{Distance between point and plane} = \frac{\left|\overrightarrow{PQ} \cdot \mathbf{n}\right|}{\|\mathbf{n}\|}$$

Equation of Plane: $ax + by + cz + d = 0$

Normal vector $\mathbf{n} = \langle a, b, c \rangle$

Surfaces in Space:

$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ Ellipsoid

$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$ Hyperboloid of One Sheet
Axis of the hyperboloid is z-axis because coefficient the variable z is negative.

$\frac{z^2}{c^2} - \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ Hyperboloid of Two Sheet
Axis of the hyperboloid is z-axis because coefficient the variable z is positive.

$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 0$ Elliptic Cone
Axis of the cone is z-axis because coefficient the variable z is negative.

$z = \frac{x^2}{a^2} + \frac{y^2}{b^2}$ Elliptic Paraboloid
Axis of the paraboloid is z-axis because the variable z is raised to the first power.

$z = \frac{x^2}{a^2} - \frac{y^2}{b^2}$ Hyperbolic Paraboloid
Axis of the paraboloid is z-axis because the variable z is raised to the first power.

Cylindrical and Spherical Systems

Relationship Between Cylindrical System and Rectangular System

Cylindrical System: (r, θ, z)

Rectangular System: (x, y, z)

$$x = r\cos\theta; \qquad r = r\sin\theta; \qquad z = z; \qquad x^2 + y^2 = r^2; \qquad \tan\theta = \frac{y}{x}$$

Relationship Between Spherical System and Rectangular System

Spherical System: (ρ, θ, ϕ)

Rectangular System: (x, y, z)

$$x = \rho\sin\phi\cos\theta; \qquad y = \rho\sin\phi\sin\theta; \qquad z = \rho\cos\phi;$$

$$\rho^2 = x^2 + y^2 + z^2; \qquad \tan\theta = \frac{y}{x}; \qquad \phi = \cos^{-1}\left(\frac{z}{\sqrt{x^2 + y^2 + z^2}}\right)$$

Relationship Between Spherical System and Cylindrical System

Cylindrical System: (ρ, θ, ϕ) and Spherical System: (r, θ, z)

$$r^2 = \rho^2\sin^2\phi; \qquad z = \rho\cos\phi$$

$$\rho = \sqrt{r^2 + z^2}; \qquad \phi = \cos^{-1}\left(\frac{z}{\sqrt{r^2 + z^2}}\right)$$