Multiplication Rule of Counting, Permutation and Combination

Multiplication Rule of Counting

Example 1:

Suppose we have three shirts (S1, S2, S3), 4 pairs of pants (P1, P2, P3, P4), and 3 pairs of shoes (SH1, SH2, SH3).

We want to select one shirt, one pair of pants, and one pair of shoes. How many different combinations are there?

Solution:

<table>
<thead>
<tr>
<th>S1</th>
<th>P1</th>
<th>SH1</th>
<th>S2</th>
<th>P1</th>
<th>SH1</th>
<th>S3</th>
<th>P1</th>
<th>SH1</th>
</tr>
</thead>
<tbody>
<tr>
<td>S1</td>
<td>P2</td>
<td>SH2</td>
<td>S2</td>
<td>P2</td>
<td>SH2</td>
<td>S3</td>
<td>P2</td>
<td>SH2</td>
</tr>
<tr>
<td>S1</td>
<td>P3</td>
<td>SH3</td>
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<td>P3</td>
<td>SH3</td>
<td>S3</td>
<td>P3</td>
<td>SH3</td>
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<tr>
<td>S1</td>
<td>P4</td>
<td>SH1</td>
<td>S2</td>
<td>P4</td>
<td>SH1</td>
<td>S3</td>
<td>P4</td>
<td>SH1</td>
</tr>
<tr>
<td>S1</td>
<td>P1</td>
<td>SH2</td>
<td>S2</td>
<td>P1</td>
<td>SH2</td>
<td>S3</td>
<td>P1</td>
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<td>S1</td>
<td>P3</td>
<td>SH1</td>
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<td>P3</td>
<td>SH1</td>
<td>S3</td>
<td>P3</td>
<td>SH1</td>
</tr>
<tr>
<td>S1</td>
<td>P4</td>
<td>SH2</td>
<td>S2</td>
<td>P4</td>
<td>SH2</td>
<td>S3</td>
<td>P4</td>
<td>SH2</td>
</tr>
<tr>
<td>S1</td>
<td>P1</td>
<td>SH3</td>
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<td>P1</td>
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<tr>
<td>S1</td>
<td>P2</td>
<td>SH1</td>
<td>S2</td>
<td>P2</td>
<td>SH1</td>
<td>S3</td>
<td>P2</td>
<td>SH1</td>
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<tr>
<td>S1</td>
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<tr>
<td>S1</td>
<td>P4</td>
<td>SH3</td>
<td>S2</td>
<td>P4</td>
<td>SH3</td>
<td>S3</td>
<td>P4</td>
<td>SH3</td>
</tr>
</tbody>
</table>
There are 36 combinations of one shirt, one pair of pants, and one pair of shoes.

Note: $36 = 3 \times 4 \times 3$

There are 26 letters in the alphabet.


We want to form a password of six characters.

How many arrangements of six characters can we form? Repetition is not allowed.

(Repetition not allowed means none of the six characters can be repeated.)

Solution:

\[
\begin{align*}
\text{Number of ways} \text{ for 1st character} & \times \text{Number of ways} \text{ for 2nd character} & \times \text{Number of ways} \text{ for 3rd character} & \times \text{Number of ways} \text{ for 4th character} & \times \text{Number of ways} \text{ for 5th character} & \times \text{Number of ways} \text{ for 6th character} \\
26 & \times 25 & \times 24 & \times 23 & \times 22 & \times 21 \\
= 165,765,600
\end{align*}
\]
There are 26 letters in the alphabet.


We want to form a password of six characters.

How many arrangements of six characters can we form? Repetition is allowed.

**Solution:**

\[
\begin{array}{cccccc}
& \text{Number of ways} & \text{for 1st character} & \times & \text{Number of ways} & \text{for 2nd character} \\
& 26 & 26 & \times & 26 & 26 \\
& \times & 26 & \times & 26 & 26 \\
& \times & \times & \times & 26 & 26 \\
& \text{Number of ways} & \text{for 3rd character} & \times & \text{Number of ways} & \text{for 4th character} \\
& 26 & 26 & \times & 26 & 26 \\
& \times & \times & \times & 26 & 26 \\
& \text{Number of ways} & \text{for 5th character} & \times & \text{Number of ways} & \text{for 6th character} \\
& 26 & 26 & \times & 26 & 26 \\
& \times & \times & \times & 26 & 26 \\
& 26^6 = 308,915,776
\end{array}
\]
Factorial

0! = 1
1! = 1

2! = 2 \cdot 1 = 2
3! = 3 \cdot 2 \cdot 1 = 6
4! = 4 \cdot 3 \cdot 2 \cdot 1 = 24
5! = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120
6! = 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 720
7! = 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 5040

n! = n \cdot (n - 1) \cdot (n - 2) \cdot (n - 3) \ldots \cdot 3 \cdot 2 \cdot 1

Simplify \( \frac{6!}{3!} \).

Solution:

\[
\frac{6!}{3!} = \frac{720}{6} = 120
\]
Simplify \( \frac{6!}{3!(6-3)!} \).

Solution:
\[
\frac{6!}{3!(6-3)!} = \frac{720}{(6)(6)} = \frac{720}{36} = 20
\]

Simplify \( \frac{7!}{(4!)(0!)} \).

Solution:
\[
\frac{7!}{(4!)(0!)} = \frac{5040}{(24)(1)} = \frac{5040}{24} = 210
\]
Permutation

There 26 letters in the alphabet.


We want to form a password of six characters.

How many arrangements of six characters can we form? Repetition is not allowed.

(Repetition not allowed means none of the six characters can be repeated.)

Solution:

\[
\begin{align*}
\text{Number of ways for 1st character} & \quad \text{Number of ways for 2nd character} & \quad \text{Number of ways for 3rd character} & \quad \text{Number of ways for 4th character} & \quad \text{Number of ways for 5th character} & \quad \text{Number of ways for 6th character} \\
26 & \quad \times & \quad 25 & \quad \times & \quad 24 & \quad \times \quad 23 & \quad \times \quad 22 & \quad \times \quad 21 \\
=165,765,600
\end{align*}
\]
Note that $26 \times 25 \times 24 \times 23 \times 22 \times 21 = 165,765,600$.

Also,

$$
\frac{26!}{(26-6)!} = \frac{26 \times 25 \times 24 \times 23 \times 22 \times 21 \times 20 \times 19 \times 18 \times 17 \times 1 \times 20 \times 19 \times 18 \times 17 \times 1}{20 \times 19 \times 18 \times 17 \times 1} \\
= 26 \times 25 \times 24 \times 23 \times 22 \times 21 \\
= 165,765,600
$$

For this problem, there are 26 distinct objects to select from; and 6 objects are chosen at a time (where repetition is not allowed).

Then the number of arrangements that can be formed is $\frac{26!}{(26-6)!}$.

In general, if there are $n$ distinct objects to select from; and $r$ objects are chosen at a time (where repetition is not allowed).

Then the number of arrangements that can be formed is $\frac{n!}{(n-r)!}$.

Notation for $\frac{n!}{(n-r)!}$ is $nP_r$

$nP_r$ is the number of permutations of $n$ distinct objects taken $r$ at a time.
There 26 letters in the alphabet and ten digits.

0, 1, 2, 3, 4, 5, 6, 7, 8, 9

Suppose we want to form a password with 10 characters where repetition is not allowed.

How many different arrangements of 10 characters are there.

Solution:

\[ n = 36 \]
\[ r = 10 \]

Number of arrangements that can be formed is:

\[ \frac{n!}{(n-r)!} = \frac{36!}{(36-10)!} = \frac{36!}{(26)!} \]
\[ = \frac{36 \times 35 \times 34 \times 33 \times 32 \times 31 \times 30 \times 29 \times 28 \times 27 \times 26 \times \ldots \times 3 \times 2 \times 1}{26 \times 25 \times 24 \times \ldots \times 2 \times 1} \]
\[ = 36 \times 35 \times 34 \times 33 \times 32 \times 31 \times 30 \times 29 \times 28 \times 27 \]
\[ = 922,393,263,052,800 \]
There 26 letters in the alphabet and ten digits.

A, B, C, D, E, F, G, H, I, J, K, L, M, N, O, P, Q, R, S, T, U, V, W, X, Y, Z, 0, 1, 2, 3, 4, 5, 6, 7, 8, 9

Suppose we want to form a password with 10 characters where repetition is allowed.

How many different arrangements of 10 characters are there.

Solution:

Number of arrangements that can be formed is:

\[ 36 \times 36 \times 36 \times 36 \times 36 \times 36 \times 36 \times 36 \times 36 \times 36 = (36)^{10} = 3,656,158,440,000,000 \]
From the set of 4 letters (A, B, C, D)

Suppose we want to form a string of 3 characters where repetition is NOT allowed.

How many different arrangements of 3 characters are there.

Solution:

\[ n = 4 \]
\[ r = 3 \]

Number of arrangements that can be formed is:

\[
nPr = \frac{n!}{(n-r)!} = \frac{4!}{(4-3)!} = \frac{4!}{1!} = \frac{24}{1} = 24
\]

<table>
<thead>
<tr>
<th>A, B, C</th>
<th>C, A, B</th>
</tr>
</thead>
<tbody>
<tr>
<td>A, B, D</td>
<td>C, A, D</td>
</tr>
<tr>
<td>A, C, D</td>
<td>C, B, A</td>
</tr>
<tr>
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<td>C, B, D</td>
</tr>
<tr>
<td>A, D, B</td>
<td>C, D, A</td>
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<tr>
<td>A, D, C</td>
<td>C, D, B</td>
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<tr>
<td>B, A, C</td>
<td>D, A, B</td>
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<td>B, A, D</td>
<td>D, A, C</td>
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<td>B, C, A</td>
<td>D, B, A</td>
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<td>B, C, D</td>
<td>D, B, C</td>
</tr>
<tr>
<td>B, D, A</td>
<td>D, C, A</td>
</tr>
<tr>
<td>B, D, C</td>
<td>D, C, B</td>
</tr>
</tbody>
</table>
Combination

From the set of 4 letters (A, B, C, D)

Suppose we want to form a string of 3 characters where repetition is NOT allowed.

How many different arrangements of 3 characters are there if order does not matter.


are considered the same.

Solution:

<table>
<thead>
<tr>
<th>A, B, C</th>
<th>C, A, B</th>
</tr>
</thead>
<tbody>
<tr>
<td>A, B, D</td>
<td>C, A, D</td>
</tr>
<tr>
<td>A, C, D</td>
<td>C, B, A</td>
</tr>
<tr>
<td>A, C, B</td>
<td>C, B, D</td>
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<tr>
<td>A, D, B</td>
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<td>C, D, B</td>
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<tr>
<td>B, A, C</td>
<td>D, A, B</td>
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<tr>
<td>B, A, D</td>
<td>D, A, C</td>
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<tr>
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<td>D, B, A</td>
</tr>
<tr>
<td>B, C, D</td>
<td>D, B, C</td>
</tr>
<tr>
<td>B, D, A</td>
<td>D, C, A</td>
</tr>
<tr>
<td>B, D, C</td>
<td>D, C, B</td>
</tr>
</tbody>
</table>

If order does not matter, then

b) The arrangements \(A, B, D\), \(A, D, B\), \(B, A, D\), \(B, D, A\), \(D, A, B\), \(D, B, A\) are considered the same.

c) The arrangements \(A, C, D\), \(A, D, C\), \(C, A, D\), \(C, D, A\), \(D, A, C\), \(D, C, A\) are considered the same.

d) The arrangements \(B, C, D\), \(B, D, C\), \(C, B, D\), \(C, D, B\), \(D, B, C\), \(D, C, B\) are considered the same.

Thus, if order does not matter, from the set of 4 letters \((A, B, C, D)\), there are 4 combinations of 3 characters where repetition is NOT allowed.

For this example, there are 4 objects to select from; and 3 objects are taken at a time. Hence, if we let \(n = 4\) and \(r = 3\), then the number of combinations that can be formed is:

\[
\frac{n!}{r!(n-r)!} = \frac{4!}{3!(4-3)!} = \frac{4!}{(3!)(1!)} = \frac{24}{(6)(1)} = \frac{24}{6} = 4
\]

where order of each arrangement does not matter.

Notation for \(n\) objects taken \(r\) at a time and order does not matter is:

\[
^nC_r = \frac{n!}{r!(n-r)!}.
\]
Permutation vs. Combination

There 26 letters in the alphabet and ten digits.
A, B, C, D, E, F, G, H, I, J, K, L, M, N, O, P, Q, R, S, T, U, V, W, X, Y, Z, 0, 1, 2, 3, 4, 5, 6, 7, 8, 9

**Repetition is allowed:**

Suppose we want to form a password with 10 characters.

How many different arrangements of 10 characters are there.

Solution:

Number of arrangements that can be formed is:
36 × 36 × 36 × 36 × 36 × 36 × 36 × 36 × 36 × 36
= \((36)^{10}\) = 3,656,158,440,000,000

**Repetition is NOT allowed and order does matter:**

Suppose we want to form a password with 10 characters.

How many different arrangements of 10 characters are there.

Solution:

\[ n = 36 \]
\[ r = 10 \]

Number of arrangements that can be formed is:
\[ \binom{n}{r} = \frac{n!}{(n-r)!} = \frac{36!}{(36-10)!} = \frac{36!}{(26)!} \]
\[ = \frac{36 \times 35 \times 34 \times 33 \times 32 \times 31 \times 30 \times 29 \times 28 \times 27 \times 26 \times \ldots \times 3 \times 2 \times 1}{26 \times 25 \times 24 \times \ldots \times 2 \times 1} \]
\[ = 36 \times 35 \times 34 \times 33 \times 32 \times 31 \times 30 \times 29 \times 28 \times 27 \]
\[ = 922,393,263,052,800 \]

Repetition is NOT allowed and order does NOT matter:

are the same.
\( n = 36 \)
\( r = 10 \)

Number of arrangements that can be formed is:

\[
\binom{n}{r} = \frac{n!}{r!(n-r)!} = \frac{36!}{10!(36-10)!} = \frac{36!}{10!(26)!}
\]

\[
= \frac{36 \times 35 \times 34 \times 33 \times 32 \times 31 \times 30 \times 29 \times 28 \times 27 \times 26 \times \ldots \times 3 \times 2 \times 1}{(10 \times 9 \times 8 \times \ldots \times 2 \times 1)(26 \times 25 \times 24 \times \ldots \times 2 \times 1)}
\]

\[
= \frac{36 \times 35 \times 34 \times 33 \times 32 \times 31 \times 30 \times 29 \times 28 \times 27}{(10 \times 9 \times 8 \times \ldots \times 2 \times 1)}
\]

\[
= 254,186,856
\]
Example:

The Illinois State Lottery contains 52 numbers and 6 are drawn.

If a person can win the lottery by matching the six drawn numbers in any order. Meaning, order does not matter.

If a person purchase one lottery ticket, what is the probability that he/she will win the lottery?

Solution:

Since order does not matter, the number of combinations of 6 numbers are:

\[ n = 52 \]
\[ r = 6 \]

Number of combinations that can be formed is:

\[
\binom{n}{r} = \frac{n!}{r!(n-r)!} = \frac{52!}{6!(52-6)!} = \frac{36!}{6!(46)!}
\]

\[ = 20,358,520 \]

Hence there are 20,358,520 combinations of six numbers.

Probability of winning lottery = \( \frac{1}{20,358,520} = 4.91\text{E}-08 = 0.0000000491 \)